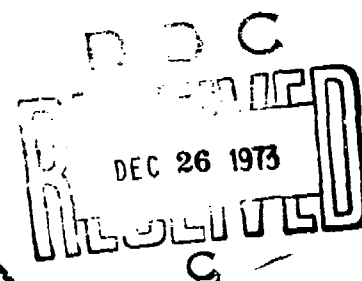AD 771544

STEREOGRAPHIC PROJECTION OF RADAR DATA
IN A NETTED RADAR SYSTEM

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The processing of data from a netted radar system may conveniently be described
in terms of the "conversion" of radar data into rectangular coordinates in a plane
with the radar site as origin and the "transformation" of these coordinates into
rectangular coordinates in the plane of a common system of coordinates. This
report focuses on the transformation aspect of the stereographic projection and com-
pletely derives the transformation equation both in complex notation and in terms that

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are suitable for real time computation. Contours of constant distortion induced by the projection process are presented. The effect of the choice of an earth's radius upon the distortion is quantitatively examined. Finally, the complete stereographic conversion and transformation equations are summarized.

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SECTION I

INTRODUCTION

In netted air defense and air traffic control systems, data from the long range radars are routed to a Sector Operations Center and stereographically projected onto a common coordinate plane for presentation to system operators on the display consoles. In this manner, the overlap coverage of the radars is exploited and a composite air surveillance picture is presented.

The projection of aircraft information from system radars may be separated into two parts namely conversion and transformation for convenience in analysis. The two aspects of the problem may be defined as follows:

(a) Conversion of slant range, azimuth and height data into rectangular coordinates in a plane with the radar site as origin.

(b) Transformation of these coordinates from the radar plane into rectangular coordinates in the common coordinate plane.

Reference 3 reports on an accurate and simple solution to the conversion problem. The data of slant range, height and azimuth angle are stereographically converted to rectangular coordinates in a plane tangent to the earth at the radar site location. Assuming tangent planes have been so established at multiple radar sites, it is desired to transform the coordinates of radar data in the various radar planes into rectangular coordinates in the common coordinate plane with accuracy and simplicity. All system computations such as aircraft tracking, interceptor control and information display are performed in the common coordinate plane.

This report focuses on the transformation aspects of the projection problem. Its purpose is to supplement Reference 3 and to derive and develop the stereographic transformation equations more completely than is accomplished in References 1 and 2. In particular this report:

(a) Derives the stereographic transformation equation in complex notation.

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(b) Derives the closed form solution to the stereo-
graphic transformation equation in terms that
are suitable for real time computation.

(c) 'Derives approximations tc the closed form solu-
tion of the stereographic transformation equation
in terms that are suitable for real time computation.

(d) Depicts error curves associated with the approxima-
tions to the closed form solution of the stereographic
transformation equation.

(e) Depicts contours of constant distortion induced by
the mapping of points on the surface of the earth
ellipsoid onto the surface of conformal spheres of
varying radii.

(f) Depicts contours of constant distortion induced
by the projection of the surface of a conformal
sphere onto a common coordinate plane.

(g) Presents contours of constant distortion induced
by the overall projection of points on the sur-
face of the earth ellipsoid onto a common coordi-
nate plane.

(h) Demonstrates how the contours of constant distor-
tion are influenced by the choice of an earth's
radius and recommends a method for determining
an optimum earth radius that minimizes the dis-
tortion associated with the overall projection
process.

(i) Demonstrates the impact of tne choice of the origin
of the common coordinate system on the distortion
contours.

(j) Summarizes the stereographic conversion and trans-
formation equations.

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SECTION II

THE TRANSFORMATION EQUATION

INTRODUCTION

Appendix A derives the equation for transforming the coordinates
of data from the plane of the radar site to a common coordinate plane
in complex notation. This section of the report presents the trans-
formation equation in complex notation, explains the various terms
therein and expresses equation (1) in terms of its real and imaginary
components. Because of the computational difficulties associated
with calculating aircraft locations in the common coordinate plane
via the derived expression, several approximations that combine sim-
plicity with accuracy are also presented.

From equation (A-13) of Appendix A, the transformation equation
in complex notation is:

$$W = \frac{Ze^{-i\beta} + W_r}{1 - \frac{Z \bar{W}_r e^{-i\beta}}{4E^2}} \tag{1}$$

where:

W = $U + iV$ are the rectangular coordinates of an
aircraft with respect to the common coordinate
origin.

Z = $X + iY$ are the rectangular coordinates of an
aircraft with respect to the radar coordinate
origin.

β = is a rotation angle which makes the axes of
the Z and W, the radar and common coordinate
planes more nearly parallel.

W_r = $U_r + iV_r$ are the rectangular coordinates of a
radar with respect to the common coordinate
origin.

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$\overline{W}_r = U_r - iV_r$ is the complex conjugate of W_r.

E = the radius of a spherical earth.

With respect to the components of the W term in equation (1), the positive U axis is directed toward east and the positive V axis is directed toward north at the origin of the system. With respect to the components of the Z term in equation (1), the positive X axis is directed toward east and the positive Y axis is directed toward north at the origin of the radar coordinate system. The coordinate axes in the radar coordinate plane are shown in Figure 1.

Given the measured slant range from the radar to an aircraft, the height of an aircraft above sea level and the elevation of the radar above sea level, the stereographic ground range R, the distance from the radar to the aircraft in the radar plane, is obtained as described in Reference 3. Having obtained the ground range R, the rectangular coordinates of the aircraft with respect to the radar coordinate origin may be expressed as:

$$Z = |Z|e^{i[(\pi/2)-\theta]} = Re^{i[(\pi/2)-\theta]} = R[Sin \theta + i Cos \theta]$$

Therefore, the rectangular coordinates of Z may be written as:

$$X = R Sin \theta$$

$$Y = R Cos \theta$$

where:

R   is the projected ground range of the aircraft.

θ is the azimuth angle measured clockwise from the positive Y axis which is oriented toward north at the location of the radar site.

The angle β in equation (1) is derived in equation (A-9) of Appendix A and depends simply upon the positional coordinates of the radar site with respect to the origin of the common coordinate plane. These coordinates are defined by their latitudes L and

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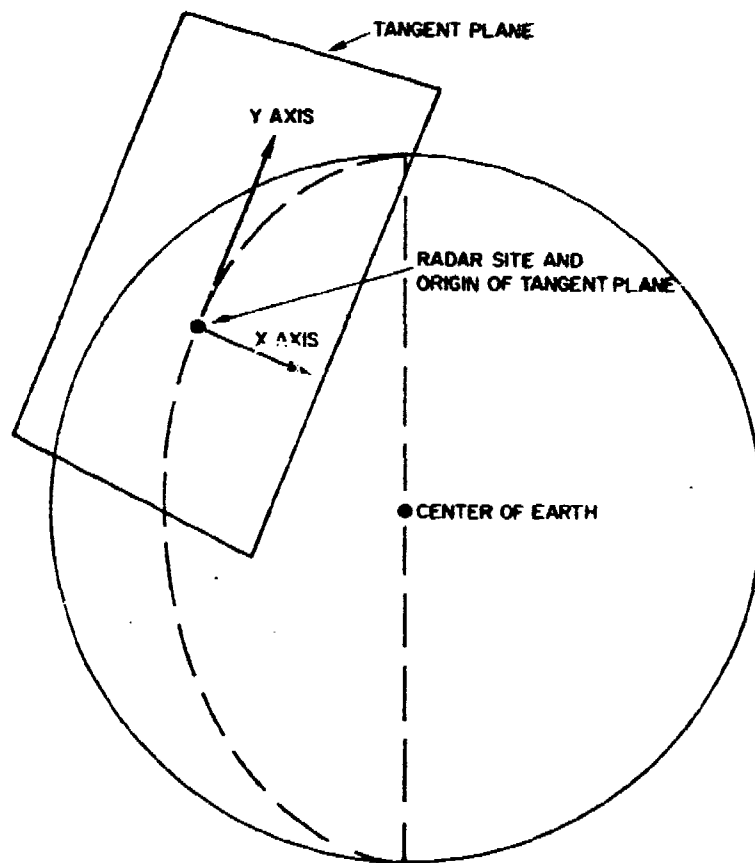
Figure 1   COORDINATE AXES IN RADAR PLANE

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longitudes λ. Latitudes north of the equator are defined as positive and latitudes south of the equator are defined as negative. Longitudes are defined as positive east of the prime meridian and negative west of the prime meridian. From equation (A-9), β, which is a constant for any radar with respect to a particular common coordinate origin, is defined as follows:

$$\beta = Tan^{-1} \frac{(Sin\ L + Sin\ L_o)\ Sin\ (\lambda_o - \lambda)}{Cos\ L\ Cos\ L_o + (1 + Sin\ L\ Sin\ L_o)\ Cos\ (\lambda - \lambda_o)} \qquad (2)$$

where:

L, λ are the latitude and longitude of the origin of the tangent plane at the radar site.

L_o, λ_o are the latitude and longitude of the origin of the common coordinate plane.

The angle β may be considered as a rotation of the Z plane with respect to the W plane. The effect of the rotation, which is counterclockwise when $\lambda < \lambda_o$ and clockwise when $\lambda > \lambda_o$, is to make the axes of the radar plane more nearly parallel to the axes of the common coordinate plane.

Considering W_r, which represents the coordinates of a radar site with respect to the common coordinate origin, Appendix B derives the equations which stereographically map points on the surface of the earth onto a rectangular coordinate plane for a spherical earth. These equations are used to project points such as the location of a radar site onto the common coordinate plane. From equation (B-5) of Appendix B, the rectangular coordinates of a radar site located at latitude L and longitude λ with respect to the common coordinate plane are:

$$U_r = 2E \frac{Sin\ (\lambda - \lambda_o)\ Cos\ L}{1 + Sin\ L\ Sin\ L_o + Cos\ L\ Cos\ L_o\ Cos\ (\lambda - \lambda_o)}$$

$$\qquad (3)$$

$$V_r = 2E \frac{Sin\ L\ Cos\ L_o - Cos\ L\ Sin\ L_o\ Cos\ (\lambda - \lambda_o)}{1 + Sin\ L\ Sin\ L_o + Cos\ L\ Cos\ L_o\ Cos\ (\lambda - \lambda_o)}$$

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where :

L,λ   are the latitude and longitude of the radar
site, the coordinates of which are projected
onto the common coordinate plane.

L_o, λ_o are the latitude and longitude of the origin
of the common coordinate plane.

U_r, V_r are the rectangular coordinates of the point
designated by L,λ in the common coordinate
plane.

    Thus far, a brief description of each of the terms in equation (1)
has been presented. Equations for obtaining the rectangular coordi-
nates of an aircraft in a plane tangent to the earth at the location
of a radar site have been presented. Assuming a common coordinate
tangent plane has been established at a location different from the
radar site, results have been derived for the rotation angle β which
make the axes of the radar plane more nearly parallel to those of
the common coordinate plane. Additionally, equations have been
derived which project the earth coordinates of a radar onto a point
in the common coordinate plane. It is now desired to employ this
information in expressing equation (1) in terms of its real and
imaginary components.

CLOSED FORM TRANSFORMATION EQUATION

    Equation (1) is the complex representation of an aircraft's posi-
tion in the common coordinate plane. Expressing the numerator of this
equation in terms of R, θ, β and $|W_r|$, we note that:

$$Ze^{-i\beta} = (X + iY)(Cos\ \beta - iSin\ \beta) = (X\ Cos\ \beta + Y\ Sin\ \beta) + i(Y\ Cos\ \beta - X\ Sin\ \beta)$$

$$= R\ Sin\ (\theta + \beta) + iR\ Cos\ (\theta + \beta) = A + iB$$

Therefore, the numerator of equation (1) is:

$$[R\ Sin\ (\theta + \beta) + |W_r|\ Sin\ \gamma] + i[R\ Cos\ (\theta + \beta) + |W_r|\ Cos\ \gamma] = C + iD$$

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The denominator of equation (1) may be expressed in terms of R, θ, β, W_r and E as follows :

$$1 - \frac{Ze^{-i\beta}\,\overline{W}_r}{4E^2} = 1 - \frac{(A + iB)(U_r - iV_r)}{4E^2} = 1 - \left[\frac{(U_rA + V_rB) + i(U_rB - V_rA)}{4E^2}\right]$$

$$= \left(1 - \frac{U_rA + V_rB}{4E^2}\right) - i\left(\frac{U_rB - V_rA}{4E^2}\right) = G - iF$$

where:

$$A = R\,Sin\,(\theta + \beta)$$

$$B = R\,Cos\,(\theta + \beta)$$

$$C = A + U_r = R\,Sin\,(\theta + \beta) + |W_r|\,Sin\,\gamma$$

$$D = B + V_r = R\,Cos\,(\theta + \beta) + |W_r|\,Cos\,\gamma$$

$$G = 1 - \frac{U_rA + V_rB}{4E^2} = \frac{4E^2 - |W_r|\,R\,Cos\,[\gamma - (\theta + \beta)]}{4E^2}$$

$$F = \frac{U_rB - V_rA}{4E^2} = \frac{|W_r|\,R\,Sin\,[\gamma - (\theta + \beta)]}{4E^2}$$

$$|W_r| = [U_r^2 + V_r^2]^{\frac{1}{2}}$$

$$\gamma = Tan^{-1}\left(U_r/V_r\right)$$

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Employing the above definitions, equation (1) may be written as:

$$W = U + iV = \frac{C + iD}{G - iF} = \frac{(GC - FD) + i(GD + FC)}{G^2 + F^2}$$

with:

$$U = \frac{GC - FD}{G^2 + F^2} \quad\text{and}\quad V = \frac{GD + FC}{G^2 + F^2}$$

Evaluating $GC - FD$, $GD + FC$, and $G^2 + F^2$ we obtain:

$$GC - FD = R \sin(\theta + \beta) + U_r - \left\{ \frac{U_r R^2 + |W_r|^2 R \sin[2\gamma - (\theta + \beta)]}{4E^2} \right\}$$

$$GD + FC = R \cos(\theta + \beta) + V_r - \left\{ \frac{V_r R^2 + |W_r|^2 R \cos[2\gamma - (\theta + \beta)]}{4E^2} \right\}$$

$$G^2 + F^2 = 1 - \frac{2|W_r| R \cos[\gamma - (\theta + \beta)]}{4E^2} + \frac{|W_r|^2 R^2}{(4E^2)^2}$$

Therefore, the closed form solution to the stereographic transformation equation is:

$$U = \frac{R \sin(\theta + \beta) + U_r - \left\{ \dfrac{U_r R^2 + |W_r|^2 R \sin[2\gamma - (\theta + \beta)]}{4E^2} \right\}}{1 - \dfrac{2|W_r| R \cos[\gamma - (\theta + \beta)]}{4E^2} + \left[\dfrac{|W_r| R}{(4E^2)} \right]^2}$$

$$V = \frac{R \cos(\theta + \beta) + V_r - \left\{ \dfrac{V_r R^2 + |W_r|^2 R \cos[2\gamma - (\theta + \beta)]}{4E^2} \right\}}{1 - \dfrac{2|W_r| R \cos[\gamma - (\theta + \beta)]}{4E^2} + \left[\dfrac{|W_r| R}{4E^2} \right]^2}$$

(4)

Because of the computational difficulties associated with calculating aircraft locations via these expressions for each radar datum, an approximation that combines suitable accuracy with minimal processing requirements is desirable.

APPROXIMATE TRANSFORMATION EQUATION

Equation (1), which for convenience is reiterated below, projects aircraft onto a tangent plane centered at the common coordinate origin.

$$W = \frac{z' + W_r}{1 - \frac{z' \overline{W}_r}{4E^2}} \tag{5}$$

where:

$$z' = Ze^{-i\beta}$$

Expanding equation (5) yields:

$$W = (z' + W_r) \left[1 - \frac{z' \overline{W}_r}{4E^2} \right]^{-1} = (z' + W_r) \left[1 + \sum_{n=1}^{\infty} \left(\frac{\overline{W}_r z'}{4E^2} \right)^n \right]$$

$$= W_r + W_r \sum_{n=1}^{\infty} \left[\frac{\overline{W}_r z'}{4E^2} \right]^n + z' + z' \sum_{n=1}^{\infty} \left[\frac{\overline{W}_r z'}{4E^2} \right]^n$$

$$= W_r + |W_r|^2 \sum_{n=1}^{\infty} \frac{(\overline{W}_r)^{n-1} (z')^n}{(4E^2)^n} + z' + \sum_{n=1}^{\infty} \frac{(\overline{W}_r)^n (z')^{n+1}}{(4E^2)^n} \tag{6}$$

$$= W_r + \frac{|W_r|^2}{4E^2} \sum_{n=0}^{\infty} \frac{(\overline{W}_r)^n (z')^{n+1}}{(4E^2)^n} + \sum_{n=0}^{\infty} \frac{(\overline{W}_r)^n (z')^{n+1}}{(4E^2)^n}$$

$$= W_r + \left[1 + \frac{|W_r|^2}{4E^2} \right] \left[\sum_{n=0}^{\infty} \frac{(\overline{W}_r)^n (z')^{n+1}}{(4E^2)^n} \right]$$

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Let θ and γ be the azimuths of Z and W_r respectively. Positive values of θ are measured clockwise from true north at the radar site and in the radar plane. Positive values of γ are measured clockwise in the common coordinate plane relative to true north at the origin of the W plane. Positive values of the rotation angle β, which is derived in Appendix A and presented in equation (2), are measured counterclockwise from east at the radar site and in the radar plane. In order that θ, γ and β may properly be combined, it is necessary to describe θ and γ in the same coordinate system as that in which β is measured. The angles corresponding to θ and γ in a system where positive angles are measured counterclockwise from east are $[(\pi/2)-\theta]$ and $[(\pi/2)-\gamma]$ respectively. Therefore, we may write:

$$(\overline{W}_r)^n = |W_r|^n e^{-in[(\pi/2) - \gamma]}$$

$$(z')^{n+1} = (Ze^{-i\beta})^{n+1} = \left\{ |z|e^{i[(\pi/2) - \theta]}e^{-i\beta} \right\}^{n+1}$$

$$= |z|^{n+1} e^{i(n+1)[(\pi/2) - (\theta + \beta)]} = |z|^{n+1} e^{i(n+1)[(\pi/2) - \theta']}$$

where:

$$\theta' = (\theta + \beta)$$

From equation (6) and employing the above definitions, W may be written as:

$$W = W_r + K \left\{ \sum_{n=0}^{\infty} \frac{|W_r|^n |z|^{n+1}}{(4E^2)^n} e^{-in[(\pi/2) - \gamma]} e^{i(n+1)[(\pi/2) - \theta']} \right\}$$

$$= W_r + K \left\{ \sum_{n=0}^{\infty} \frac{|W_r|^n |z|^{n+1}}{(4E^2)^n} e^{i[(\pi/2) - \{\theta' + n(\theta' - \gamma)\}]} \right\}$$

$$= W_r + K \sum_{n=0}^{\infty} \frac{|W_r|^n |z|^{n+1}}{(4E^2)^n} \left\{ \mathrm{Sin}\,[(\theta+\beta)+n(\theta+\beta-\gamma)] + i\,\mathrm{Cos}\,[(\theta+\beta)+n(\theta+\beta-\gamma)] \right\}$$

Therefore, the components of W are:

$$U = U_r + K \sum_{n=0}^{\infty} \frac{|W_r|^n |z|^{n+1}}{(4E^2)^n} \quad \text{Sin} \left[(\theta + \beta) + n (\theta + \beta - \gamma) \right]$$

$$V = V_r + K \sum_{n=0}^{\infty} \frac{|W_r|^n |z|^{n+1}}{(4E^2)^n} \quad \text{Cos} \left[(\theta + \beta) + n (\theta + \beta - \gamma) \right]$$

(7)

where:

$$K = 1 + \frac{|W_r|^2}{4E^2}$$

Approximations to U and V may be obtained by terminating this series solution to equation (1) after an appropriate number of terms.

First Order Approximation

The first order approximation to equation (1) is obtained by taking the first term of equation (7) as follows:

$$U = U_r + K |z| \text{ Sin } (\theta + \beta) = U_r + K R \text{ Sin } (\theta + \beta)$$

$$V = V_r + K |z| \text{ Cos } (\theta + \beta) = V_r + K R \text{ Cos } (\theta + \beta)$$

(8)

The error in this approximation to equation (1) is on the order of the first term neglected. Values for the first term neglected, namely $K|W_r|R^2/4E^2$, are presented in Figure 2 for ground ranges from 25 to 200 miles and for $|W_r|$ from 0 to 550 miles.

Figure 2 shows the transformation errors associated with a first order approximation to equation (1). These errors exceed 0.4 miles at large values of R and $|W_r|$ and approach 0.25 miles at intermediate values of R and $|W_r|$. The composite projection error is the sum of the transformation error as shown in Figure 2 and the conversion error as shown in Reference 3.
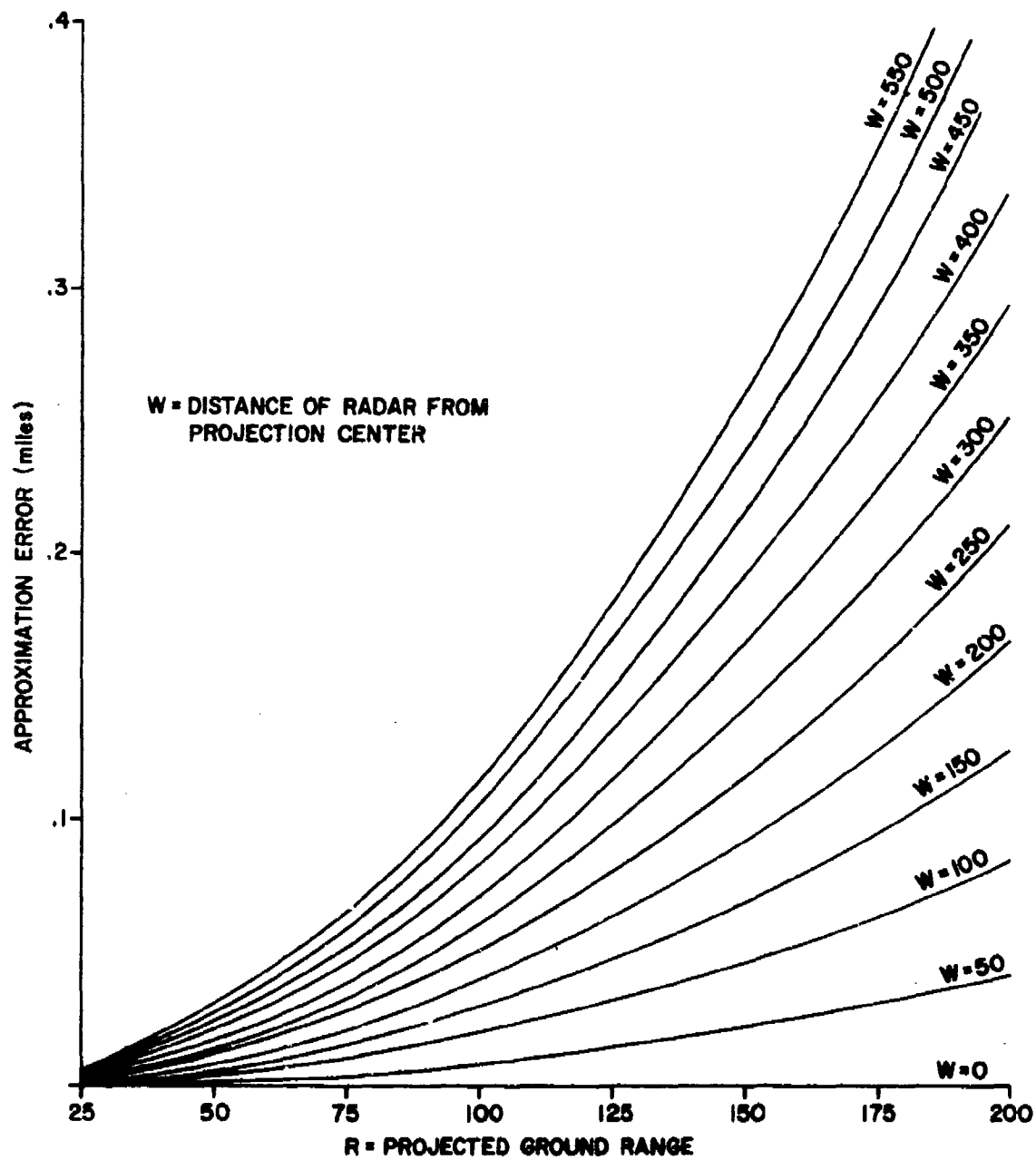
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Figure 2   ERROR USING FIRST ORDER APPROXIMATION

When aircraft height information is available, the upper limit on the conversion error is expected to be 0.1 miles. This result has been obtained by consideration of Figure 6 of Reference 3. Assuming a maximum distance between a radar site and the projection center of 275 miles, the maximum transformation error from Figure 2 herein is anticipated to be 0.25 miles using a first order approximation. The composite maximum error is 0.35 miles, which is deemed unacceptable. Consequently, when aircraft height information is available, a higher order approximation to equation (1) that will essentially decrease the transformation error to zero as will be shown hereafter, is required.

From Figure 12 of Reference 3, it is noted that errors in excess of one mile occur in converting slant range to ground range when aircraft height data is unavailable. The fundamental remedy to this problem is to obtain height information on the aircraft and thereby diminish the relatively large conversion error rather than attempting to decrease a first order transformation error that may at most equal 0.25 miles. Therefore, when aircraft height information is unavailable, a first order approximation to equation (1) is acceptable.

Second Order Approximation

The second order approximation to equation (1) is obtained by taking the first two terms of equation (7) as follows:

$$U = U_r + K \left\{ R \sin (\theta + \beta) + \frac{R^2 |W_r| \sin [2(\theta + \beta) - \gamma]}{4E^2} \right\}$$

$$V = V_r + K \left\{ R \cos (\theta + \beta) + \frac{R^2 |W_r| \cos [2(\theta + \beta) - \gamma]}{4E^2} \right\},$$

(9)

The error in this approximation is on the order of the third term of the infinite series. Values for the third term, namely $K|W_r|^2 R^3 / 16E^4$ are presented in Figure 3 for ground ranges from 75 to 200 miles and for $|W_r|$ from 0 to 550 miles. Examination of Figure 3 shows that the second order approximation to equation (1) essentially yields identical results to the closed form solution to the stereographic transformation equation.
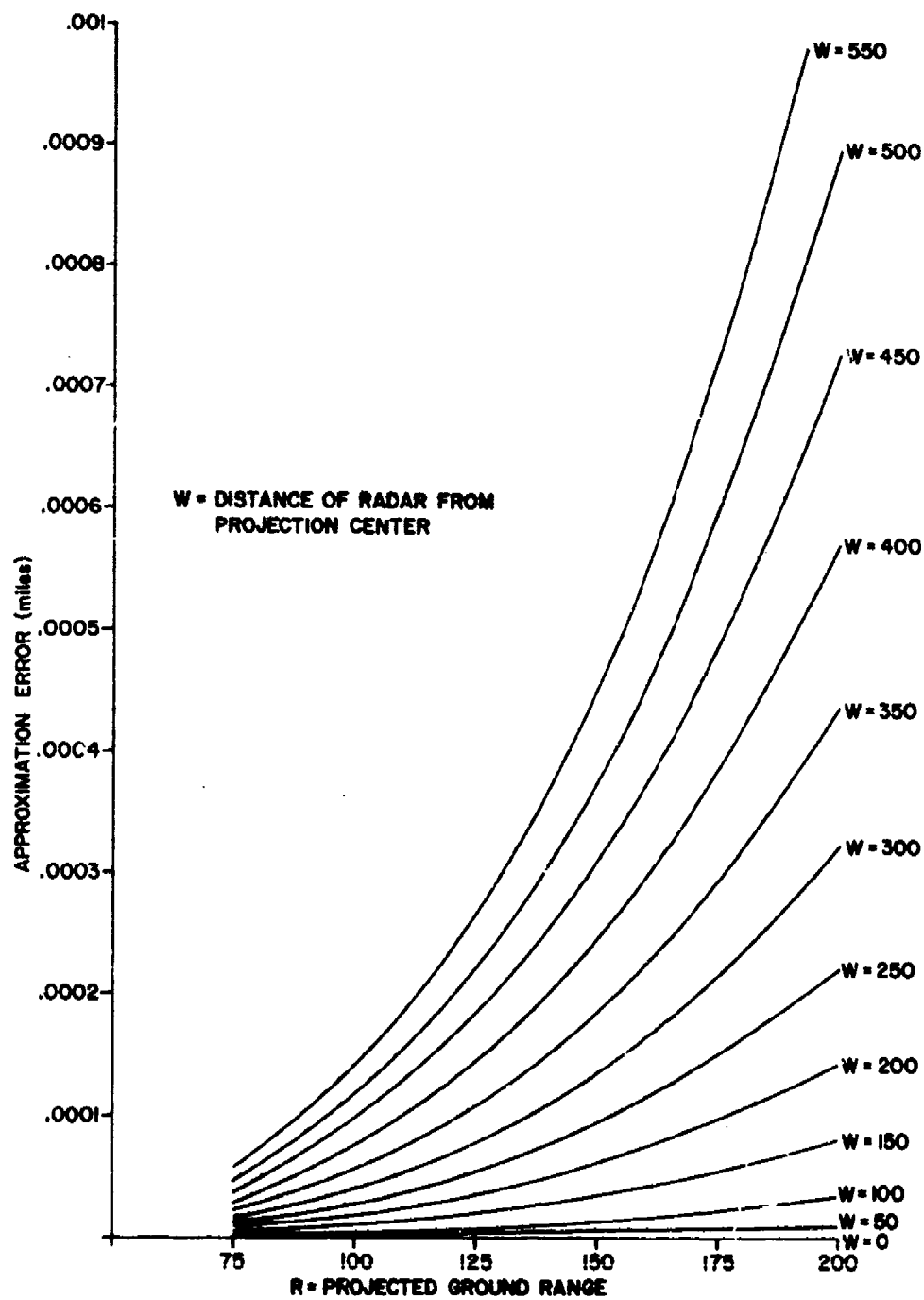
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Figure 3   ERROR USING SECOND ORDER APPROXIMATION

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For programming convenience, equations (9) may be further simplified by a numerical analysis of the second order range term. When height data is available a suitable expression for the ground range R, which is derived in equation (8) of Reference 3, is given by:

$$R = \frac{[S^2 - (H-h)^2]^{\frac{1}{2}}}{1 + \frac{2}{E}}$$

where:

S   is the measured slant range of the aircraft.

H   is the elevation of the aircraft above sea level.

h   is the elevation of the radar above sea level.

E   is the radius of a spherical earth.

Substituting this result into the U component portion of equation (9) yields:

$$U = U_r + K \left\{ \frac{[S^2-(H-h)^2]^{\frac{1}{2}}}{1 + \frac{2}{E}} \sin(\theta+\beta) + \frac{[S^2-(H-h)^2]}{(1 + \frac{2}{E})^2} \frac{|W_r| \sin[2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

Calculating the maximum magnitude of the $\{|W_r| \sin[2(\theta+\beta) - \gamma]\}/4E^2)$ term, we obtain the result 1.056×10^{-5} when $|W_r|$ equals 500 miles. This multiplier of the R^2 term is miniscule; consequently, the denominator of the R^2 term may be changed to $(1 + 2/E)$ while retaining an accuracy of better than 3.1×10^{-4} miles for values of R through 225 miles and for values of $|W_r|$ through 500 miles.

Rewriting equations (9) as suggested, we obtain a revised approximation to equation (1) that retains the accuracy of equations (9) while simplifying the formulation and processing requirements. The simplified second order approximation is:

$$U = U_r + T \left\{ [S^2-(H-h)^2]^{\frac{1}{2}} \sin(\theta+\beta) + \frac{[S^2-(H-h)^2] |W_r| \sin[2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

$$V = V_r + T \left\{ [S^2-(H-h)^2]^{\frac{1}{2}} \cos(\theta+\beta) + \frac{[S^2-(H-h)^2] |W_r| \cos[2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

(9a)

where:

$$T = \frac{1 + \dfrac{|W_r|^2}{4E^2}}{1 + \dfrac{2}{E}}$$

T is a site dependent constant and may be precalculated and stored as an adaptation parameter.

SECTION III

DISTORTION INDUCED BY PROJECTION

INTRODUCTION

The earth is not a sphere but is an ellipsoid that is approximately 23 miles greater in equatorial than polar diameter. This fact has been deliberately ignored until this section of the report as it appeared wiser to initially consider a spherical earth and later to consider the implications of the earth's ellipticity. A topic that must now be addressed is the value of an earth's radius that should be used in any of the equations which are dependent upon the earth's size and shape. A related topic is the distortion or error that is introduced by the entire mapping process.

Any projection of an ellipsoid onto a plane distorts either distances, angles or areas, it being impossible to simultaneously preserve all of these parameters. Furthermore, no projection of an ellipsoid onto a plane can simultaneously preserve both angles and areas. The stereographic projection preserves angles but distorts distances.

This section of the report will address the manner in which the earth's ellipticity must be taken into account. Additionally, the magnitude of the distortion induced by the projection of an ellipsoid onto a plane will be discussed. Considerations relative to the choice of an earth's radius in the transformation equations will be discussed. The effect of the choice of an earth's radius upon the overall distortion associated with stereographic projection will be quantitatively presented.

By way of introduction to these topics, the stereographic projection must be considered in greater detail.

STEREOGRAPHIC PROJECTION

The process of stereographically projecting points onto a plane is accomplished in what may be considered as two distinct steps. The surface of the earth is approximated by the surface of an ellipsoid, which is obtained by rotating an ellipse about its minor axis. The first step maps points on or above a designated point on the ellipsoid onto a sphere which will be called the conformal sphere. The second step maps points on the conformal sphere onto a plane tangent to the conformal sphere.

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The mapping of points on or above a designated location on the ellipsoid onto a sphere is accomplished by the following relations as indicated on page 34 of Reference 4:

$$\lambda' = \lambda$$

$$\text{Tan}\left[\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] = \text{Tan}\left[\left(\frac{\pi}{4} + \frac{L}{2}\right)\right] \cdot \left[\frac{1 - \epsilon \, \text{Sin} \, L}{1 + \epsilon \, \text{Sin} \, L}\right]^{\epsilon/2} \tag{10}$$

where:

L, λ are the latitude and longitude of the point on the ellipsoid.

ϕ, λ' are the latitude and longitude of the corresponding point projected onto the sphere. This latitude is defined as the conformal latitude.

ϵ = .08199189 is the eccentricity of the earth.

Therefore, the equations for mapping an ellipsoid onto a conformal sphere and for stereographically mapping the conformal sphere onto a plane are the same as those that have been presented for a spherical earth except that the conformal latitude ϕ must be used in place of the geographic latitude L. For computational convenience, the conformal latitude ϕ may be obtained from the geographic latitude L by a series approximation to equation (10).

As noted on page 34 of Reference 4, the mapping from the ellipsoid to the sphere is conformal. The mapping from the sphere to the plane is also conformal. Each step preserves angles; hence, the resultant stereographic projection is a conformal representation of the earth ellipsoid on a plane.

DISTORTION AND EARTH'S RADIUS

The surface of an ellipsoid cannot be molded into a sphere without some distortion or error. Similarly, the surface of an appropriately chosen sphere cannot be spread out flat without some distortion. In general, arc lengths that are projected onto a planar surface are stretched as the distance between the point of tangency and the point being projected increases. The magnitude of the stretching is non-linear. Therefore, groups of positional reports from the same aircraft exhibit distortions when projected on a plane. Data from a constant speed aircraft, that is constrained to move at a constant altitude above the earth's surface, portray apparent accelerations when projected onto a plane.

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Intuitively, it appears evident that the distortions described above can be minimized by a judicious choice of the earth's radius. The following paragraphs will show quantitative relationships between the choice of an earth's radius and the resultant distortion for the mapping from the ellipsoid to the conformal sphere and for the projection of the conformal sphere onto a plane.

Mapping from Ellipsoid to Sphere

The scale factor associated with the mapping of points from an ellipsoid onto a sphere is defined as the ratio of the arc length along the surface of the sphere to the corresponding arc length along the surface of the ellipsoid. A scale factor of unity indicates that distances are completely preserved and that there is no distortion in the mapping process. From page 86 of Reference 5, the scale factor k_1, associated with the projection of the ellipsoid onto the conformal sphere is:

$$k_1 = \frac{ds_1}{ds_2} = \frac{E \cos \phi}{N \cos L} \qquad (11)$$

where:

k_1 is the scale factor.

ds_1 is the arc length along the surface of the conformal sphere.

ds_2 is the arc length along the surface of the ellipsoid.

E is the radius of the conformal sphere.

L is the geographic latitude. This is the angle which a normal to the surface of an ellipsoid makes with the equatorial plane as shown in Figure 4.

ϕ is the conformal latitude of the point whose geographic latitude is L.

N is the distance MP in Figure 4 where a normal to the ellipsoid at point M has been drawn until it intersects the minor axis at P. The normal N is mathematically expressed as $Eq (1 - \epsilon^2 \sin^2 L)^{-\frac{1}{2}}$ where $Eq = 3444.054$ miles is the earth's equatorial radius and where $\epsilon^2 = .00672267$ is the earth's eccentricity squared.

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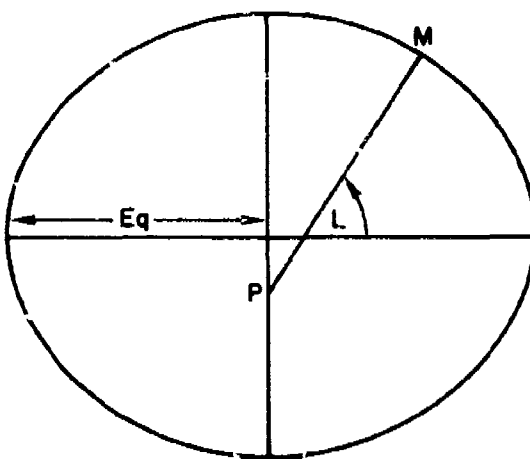
Figure 4

NORMAL TO THE ELLIPSOID

Equation (11) expresses the scale factor k_1 in terms of the earth's radius at the equator, the earth's eccentricity, the geographic latitude and the radius of the conformal sphere. Therefore, we may set the scale factor k_1 equal to unity at a designated latitude and solve for the resultant radius of the conformal sphere according to the relationship:

$$E = \frac{N \ Cos \ L}{Cos \ \phi} \tag{12}$$

Equation (12) determines the radius of the conformal sphere that will result in a unity scale factor at the latitude that is used in the computation. Table 1 shows the radii of the conformal spheres that result from solving equation (12) for latitudes L of 0, 20 and 40 degrees. The conformal latitudes ϕ that correspond to these geographic latitudes are also shown.

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Table I

Earth Radius Versus Latitude (Ellipsoid To Sphere)

Geographic Latitude	Conformal Latitude	Earth Radius (Mi)
0.0	0.00	3444.05
20.0	19.88	3442.71
40.0	39.81	3439.29

Table I shows the radii of the conformal spheres that result from solving equation (12) using the indicated geographic latitudes. When the mapping of the ellipsoid onto the sphere is centered at the equator, the optimum radius of the conformal sphere is the earth's equatorial radius of 3444.05 miles. When the mapping is centered at more northerly latitudes, the optimum earth radii decrease as shown in the table.

Let us now determine the scale factors or magnifications that result from solving equation (11) using each of the earth radii contained in Table I over a designated set of latitudes. It is evident that the radius of the conformal sphere, which is calculated based upon a particular latitude, will result in magnifications closest to unity over a geographical area that is centered at the same latitude. To quantify this judgment, Table II shows the solutions to equation (11) for latitudes between 36 and 44 degrees when E assumes each of the three values shown in Table I.

Table II

Magnification Values

Latitude	Latitude/Corresponding Radius of Conformal Sphere		
	0.0/3444.05	20.0/3442.71	40/3439.29
36	1.00116	1.00077	0.99977
38	1.00127	1.00088	0.99989
40	1.00138	1.00099	1.00000
42	1.00150	1.00111	1.00012
44	1.00162	1.00123	1.00023

Each column of Table II shows the magnification value that re-
sults from solving equation (11) using the indicated value of the
radius of the conformal sphere while varying the latitude from 36
to 44 degrees. When the radius of the conformal sphere is calculated
based upon unity magnification at latitudes of either 0 or 20 degrees,
distances on the ellipsoid over all latitudes from 36 to 44 degrees
are stretched when projected onto the respective spheres. When the
radius of the conformal sphere is based upon unity magnification at
a latitude of 40 degrees, distances on the ellipsoid at latitudes
above 40 degrees are elongated whereas distances below a latitude
of 40 degrees are contracted when projected onto the sphere. As
expected, the magnitudes of the elongations and contractions are
minuscule. Figure 5 depicts the situation when the earth's radius
is chosen based upon the latitudinal center of the geographical area
of interest, designated L. As indicated in Table II, this radius
of the conformal sphere is optimum in that it results in magnifica-
tions closest to unity over the area of interest.

Mapping from Ellipsoid to Plane

The scale factor k_2 that is associated with the process of map-
ping points on the conformal sphere onto points on the common coordi-
nate plane is from page 41 of Reference 4:

$$k_2 = \frac{2}{1 + \text{Sin } \phi \text{ Sin } \phi_o + \text{Cos } \phi \text{ Cos } \phi_o \text{ Cos } (\lambda-\lambda_o)} \tag{13}$$

where:

ϕ, λ are the conformal latitude and longitude of the
point to be projected.

ϕ_o, λ_o are the conformal latitude and longitude of the
origin of the common coordinate plane. This
origin will be referred to as the projection
center.

From equation (B-1) of Appendix B, it will be noted that the de-
nominator of equation (13) may be written as:

$$1 + \text{Sin } \phi \text{ Sin } \phi_o + \text{Cos } \phi \text{ Cos } \phi_o \text{ Cos } (\lambda-\lambda_o) = 1 + \text{Cos } D$$

where:

D is the angle that subtends the great circle arc from (ϕ, λ)
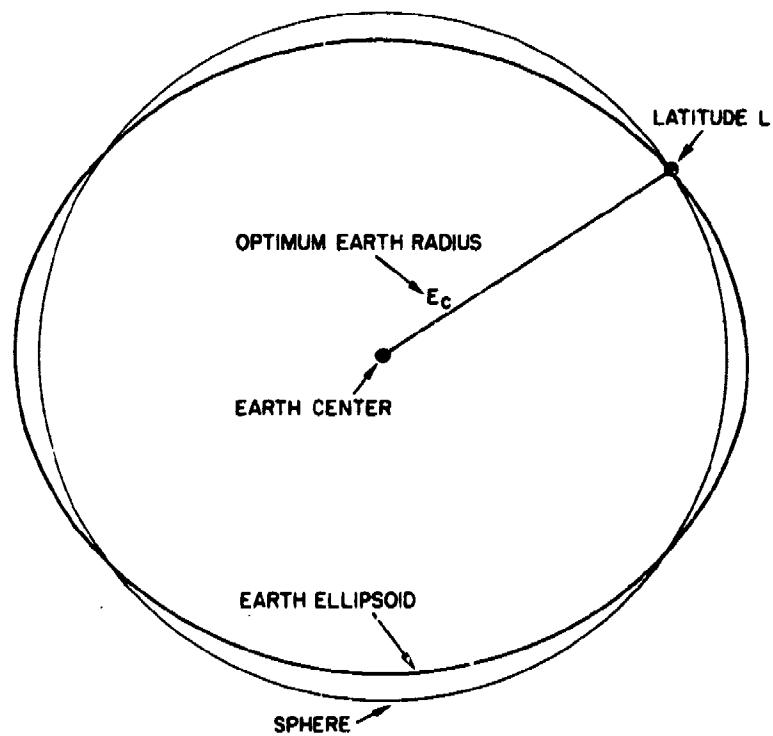to (ϕ_o, λ_o).

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Figure 5   OPTIMUM EARTH RADIUS FOR MAPPING
OF ELLIPSOID ONTO SPHERE

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The magnification k_2 equals unity at the projection center and increases as the distance from the projection center increases. Contours of equal scale magnification are concentric circles about the projection center. Figure 6 shows contours of equal magnification for several values of k_2 where the horizontal axis is a range of longitudes between -9 and +3 degrees and where the vertical axis is a range of latitudes between 36 and 44 degrees. From Figure 6, it will be observed that the scale is unity at the projection center, that the scale increases with the distance from the projection center and that contours of equivalent magnification are symmetric about the vertical axis. The projection center L_o, λ_o in Figure 6 is 40°, -3°.

The total scale factor associated with the projection of the ellipse onto the plane is the product of the scale factors associated with the two steps of the mapping process. Therefore, the total magnification designated k_t is:

$$k_t = \frac{2E \cos \phi}{N \cos L \ [1 + \sin \phi \sin \phi_o + \cos \phi \cos \phi_o \cos (\lambda - \lambda_o)]} \tag{14}$$

Table II has presented the magnification values associated with the projection of the ellipsoid onto conformal spheres with varying radii. Figure 6 has presented contours of equal magnification for the mapping from the conformal sphere onto the plane. Equation (14) provides a method for combining the magnifications associated with the two steps of the mapping process over a designated range of latitudes and longitudes for a specified radius of the conformal sphere.

For comparative purposes equation (14) has been evaluated using as the radius of the conformal sphere:

(a) The earth's equatorial radius

(b) The earth radius that is optimized for the mapping from the ellipsoid to the sphere when the mapping is latitudinally centered at 40 degrees.

Using the earth's equatorial radius as the radius of the conformal sphere, Figure 7 shows contours of equal magnification for the same range of latitudes and longitudes that were used in Figure 6. Examination of Figure 7 shows that contours of equal mangification remain symmetric about the vertical axis. Due to use of an earth's radius that is better suited to a projection centered at the equator than to a mapping centered at a latitude of 40 degrees, the contours of equal magnification are larger than is necessary. The magnification is 1.00138 at the projection center and increases away from the projection center.
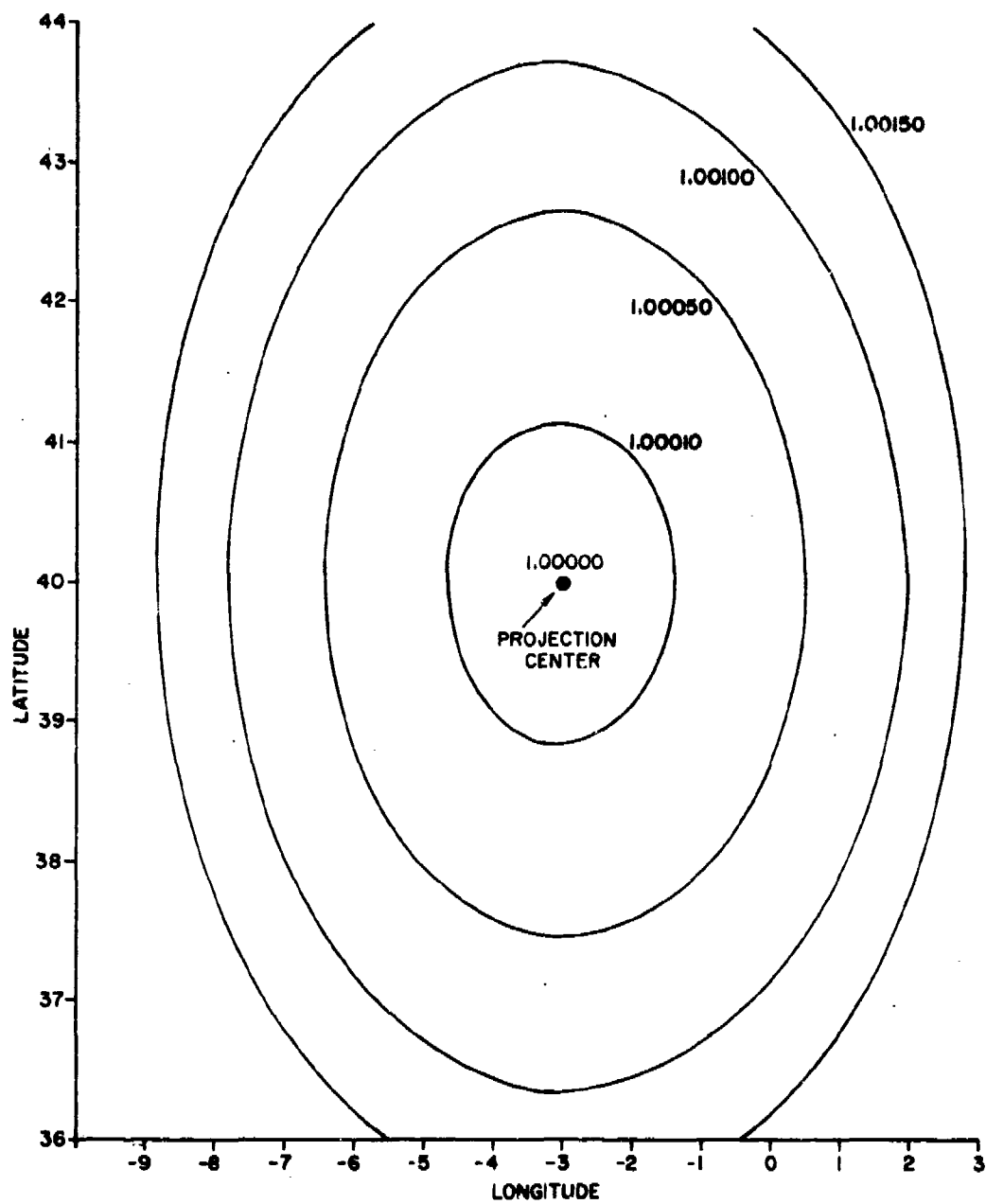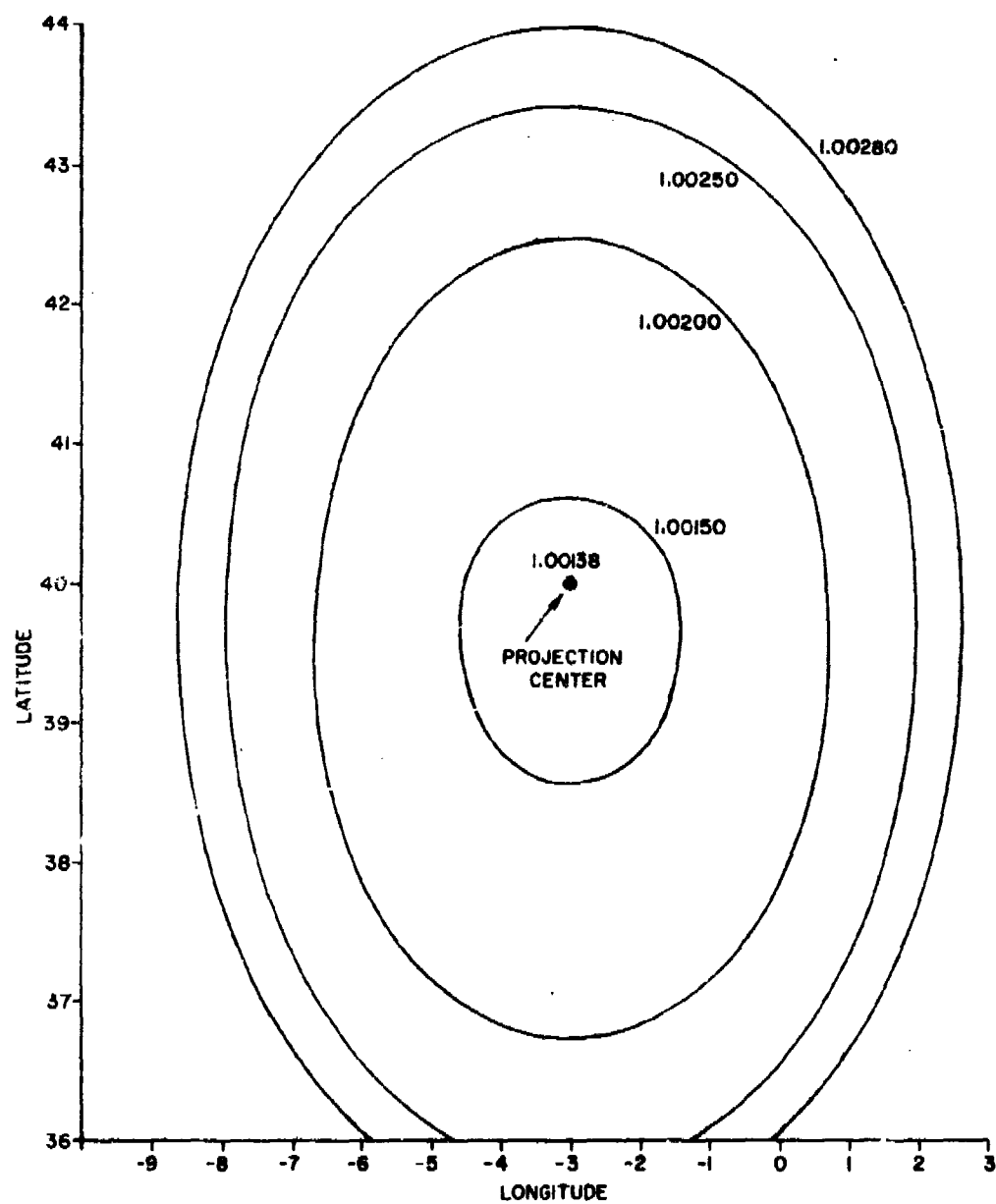
Figure 6   CONFORMAL SPHERE TO PLANE MAGNIFICATION

Figure 7   TOTAL MAGNIFICATION (E=3444.05)

Figure 8 shows contours of equal magnification using the earth's radius that is optimized for the mapping from the ellipsoid to the sphere when the mapping is latitudinally centered at 40 degrees. With cursory consideration, this choice of an earth's radius appears to minimize distance distortions over the geographical area of interest since k_1 is unity at the latitudinal center of the projection and since k_2 is completely determined for any point (L,λ) given the projection center (L_o, λ_o).

The magnification contours of Figure 8 are analogous to those of Figure 6 except that the contours of Figure 8 are no longer symmetric about the horizontal axis through the projection center; these contours are flattened to the north and elongated to the south of the projection center. Like Figure 6, the magnification is unity at the projection center, increases away from the center, and is symmetric about the vertical axis.

Through this point in the discussion, Figure 6 has presented contours of equal magnification associated with the mapping from the conformal sphere to the plane. These contours are independent of the choice of an earth radius. Figures 7 and 8 have presented contours of equal magnification based upon the overall projection from the ellipsoid to the plane. These contours are dependent upon the choice of an earth's radius and have been depicted for radii that minimize the distortion associated with the mapping from the ellipsoid to the conformal sphere over geographical areas that are latitudinally centered at 0 and 40 degrees respectively.

A common feature of Figures 6 through 8 is that the magnification is greater than or equal to unity at the projection center and increases as the distance from the projection center increases. What would appear more desirable, as it would tend to decrease the registration error between observations of the same aircraft by different radars, is a unity magnification that would occur approximately midway between the projection center and the extremity of radar coverage. This would effect magnifications of less than unity at the projection center, a contour of unity magnification within the geographic area of coverage and a magnification contour that exceeds unity at the extremity of radar coverage.

To accomplish the desired result equation (14), which for convenience is reiterated below, will be considered when setting the magnification k_t to unity at a particular value of (L,λ) and for a particular projection center (L_o,λ_o).

$$k_t = \frac{2E \cos \phi}{N \cos L \, [1 + \sin \phi \sin \phi_o + \cos \phi \cos \phi_o \cos (\lambda-\lambda_o)]} \tag{15}$$
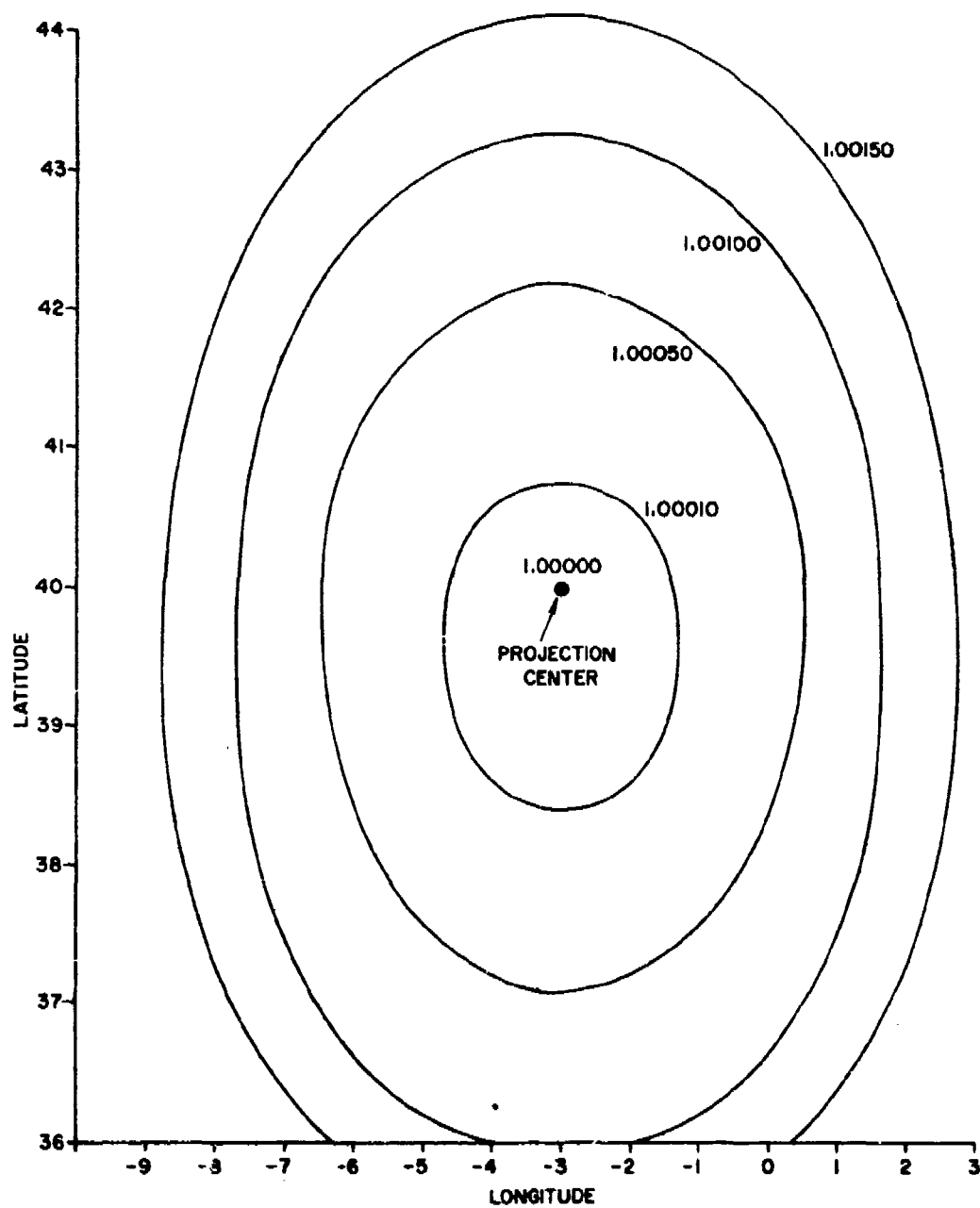
28

Figure 8   TOTAL MAGNIFICATION (E=3439.29)

Solving equation (15) for the resultant radius of the conformal sphere
E, we obtain:

$$\eta = \frac{N \cos L \left[1 + \sin \phi \sin \phi_o + \cos \phi \cos \phi_o \cos (\lambda - \lambda_o)\right]}{2 \cos \phi} \tag{16}$$

Equation (16) determines the radius of the conformal sphere that will
result in unity magnification at the point (L, λ) for the particular
projection center (L_o, λ_o). Table III snows the radii of conformal
spheres that result from solving equation (16) for several values
of (L, λ) when the projection center (L_o, λ_o) is chosen as $(40°, -3°)$.
The initial value of (L, λ), is of no practical importance since the
projection center is approximately 2400 miles away from the point
$(0°, -7°)$. Nevertheless, it has been included in Table III in order
to better understand the implications of equation (16).

<div align="center">

Table III

Radius of Conformal Sphere (Ellipsoid To Plane)

</div>

L, λ	Earth Radius
0,-7	3041.62
42,-7	3435.45
38, 0	3437.21
40,-3	3439.29

Comparing the earth radii in Table III to those in Table I, we
note that the earth radius corresponding to $(0,-7)$ is approximately
400 miles less than any of the other earth radii. When the points
(L, λ) are proximate to the projection center, the resultant earth
radii are approximately equal to the earth radius that was presented
in Table I and which was optimized for the mapping from the ellipse
to the sphere. When the point (L, λ) equals the coordinates of the
projection center, the resultant earth radius equals the earth radius
of Table I which was optimized for the mapping from the ellipse to
the sphere.

Figure 9 shows the contours of equal magnification that result
from solving equation (15) while requiring that a unity magnification
be obtained at the point $0°, -7°$. The appropriate earth radius that
was used in obtaining the contours is 3041.62 miles and the projection
center is $40°, -3°$. The geographical area over which the contours
are presented is identical to that of the previous figures.

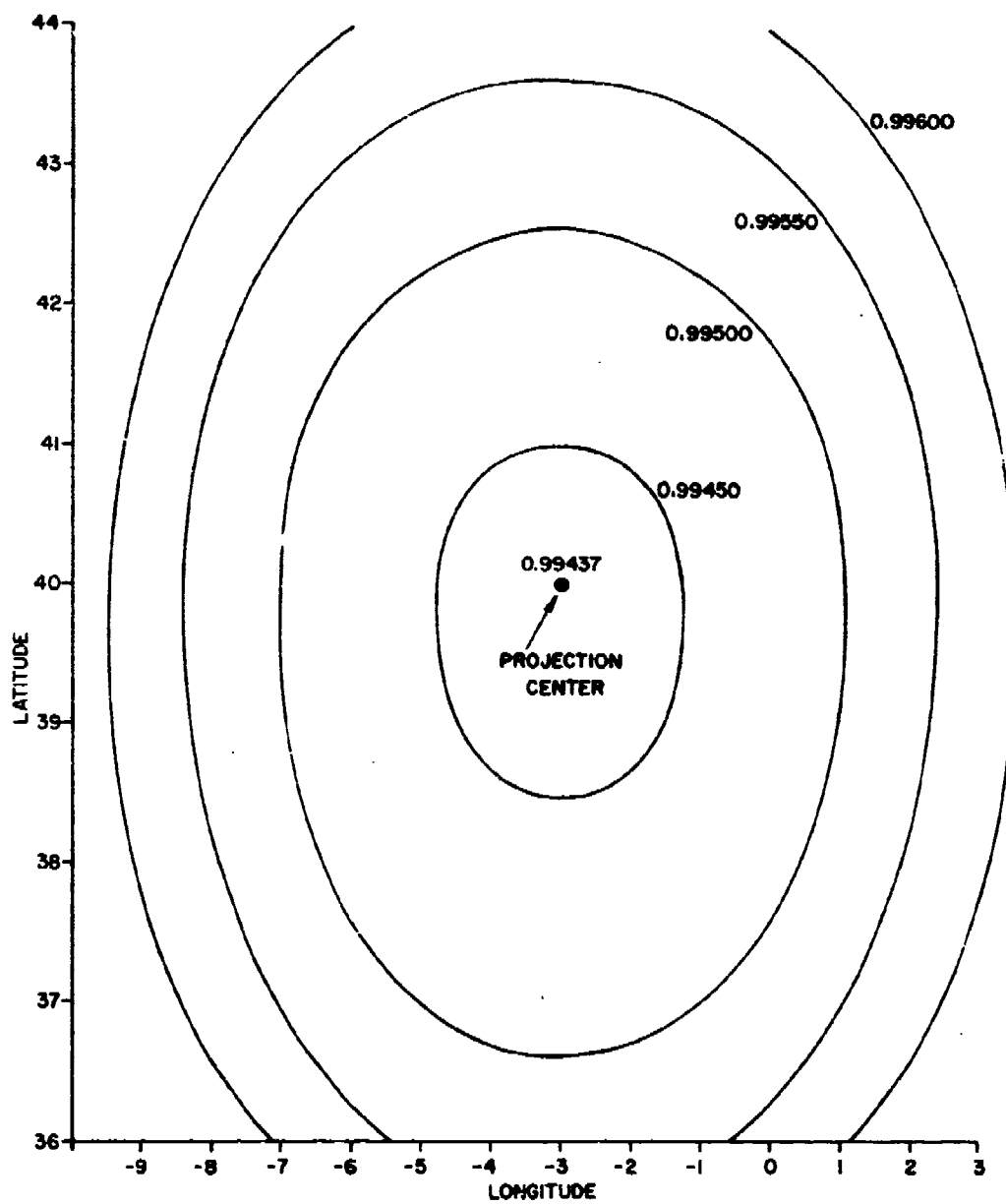Figure 9   MAGNIFICATION; L, λ = O°, −7°

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Examination of Figure 9 shows that the scale equals 0.99437 at the projection center and increases with the distance from the center. If the contours were plotted over an extended geographical area, the magnification would eventually become unity at the point $0°$, $-7°$.

Figure 10 presents contours of equal magnification when L,λ equals $42°$, $-7°$ in equation (16). Figure 11 presents similar contours when L,λ equals $38°$, $0°$. Finally, Figure 12 presents magnification contours for L,λ equal to the projection center, namely L,λ equals $40°$, $-3°$. Figure 12 is identical to Figure 6.

Figures 9 through 12 show that the magnification contours remain symmetric about the vertical axis through the specified projection center. Therefore, to minimize the distortion associated with the mapping from the conformal sphere to the common coordinate plane, the projection center should be located proximate to the center of the geographical area of interest.

Figures 9 through 12 also show that a unity magnification contour may be obtained through a designated location within the geographical area of interest by simply solving equation (16) for the appropriate radius of the conformal sphere. Using the resultant radius as the value of E in the transformation equations will effect the desired result. It is suggested that the unity magnification contour should be located approximately midway between the projection center and the maximum extent of radar coverage in order to decrease the registration error between observations of the same aircraft by different radars.

EARTH RADIUS AND COORDINATE CONVERSION

Having obtained the radius of the conformal sphere that results in a unity magnification along a particular elliptical contour, it is necessary to determine whether this radius is suitable for use in the stereographic coordinate conversion algorithms. Intuitively one could argue that the earth's radius to be used in the coordinate conversion algorithm should be obtained from equation (16) based upon the coordinates of the radar site location (L,λ) and the origin of the common coordinate plane (L_o,λ_o). Such an approach would result in a unique earth radius being associated with each radar site.

To ascertain the effect of a unique earth radius for each radar site, the extent of the variation in the earth's radius over the geographical area described by $36° \leq L \leq 44°$ and $-9° \leq \lambda \leq 3°$ was determined. Equation (16) was solved for varying (L,λ) within the geographical area with the origin of the common coordinate plane located at $L_o = 40°$ and $\lambda_o = -3°$. The resultant minimum and maximum

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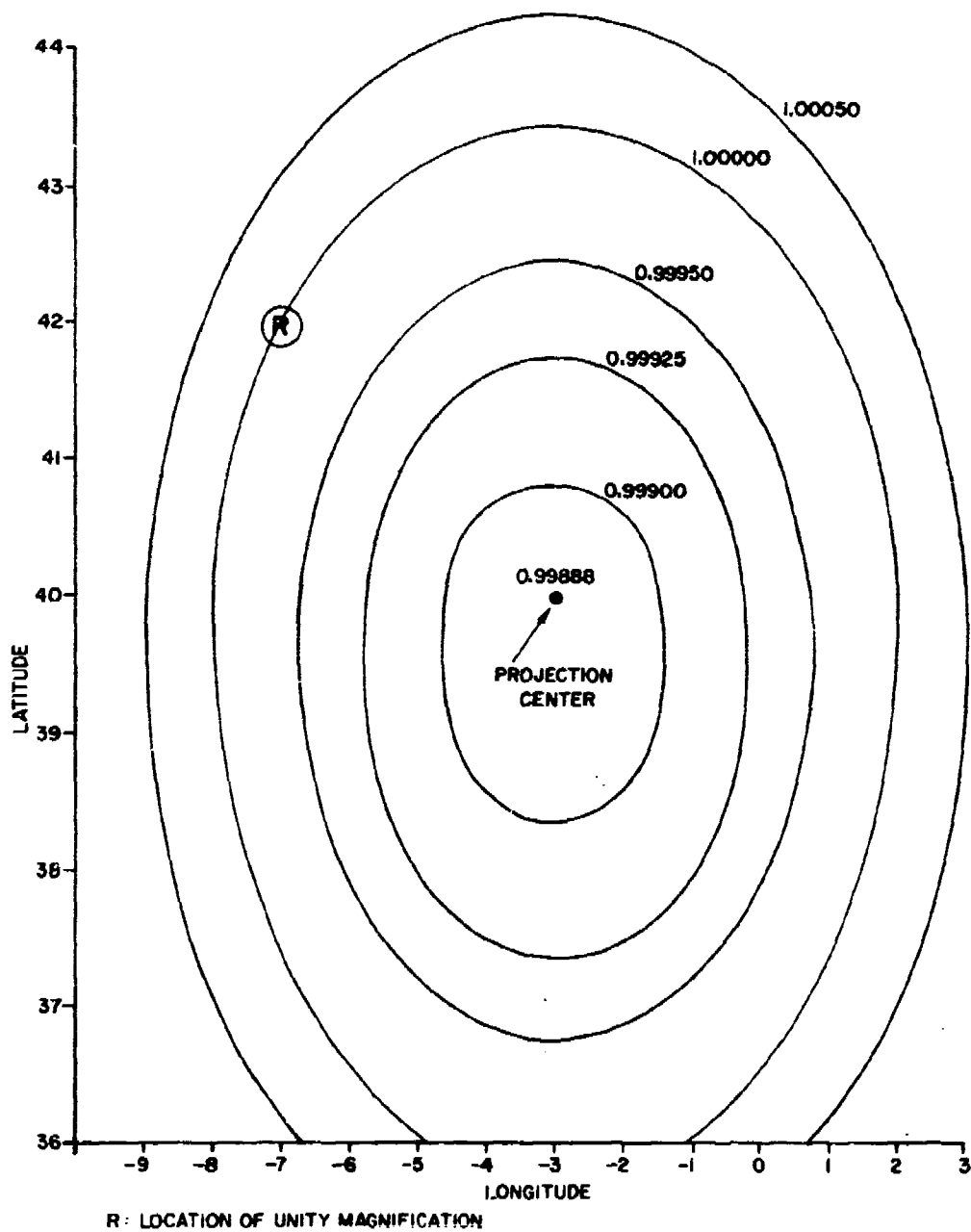Figure 10   MAGNIFICATION; L,λ = 42°,-7°

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R : LOCATION OF UNITY MAGNIFICATION
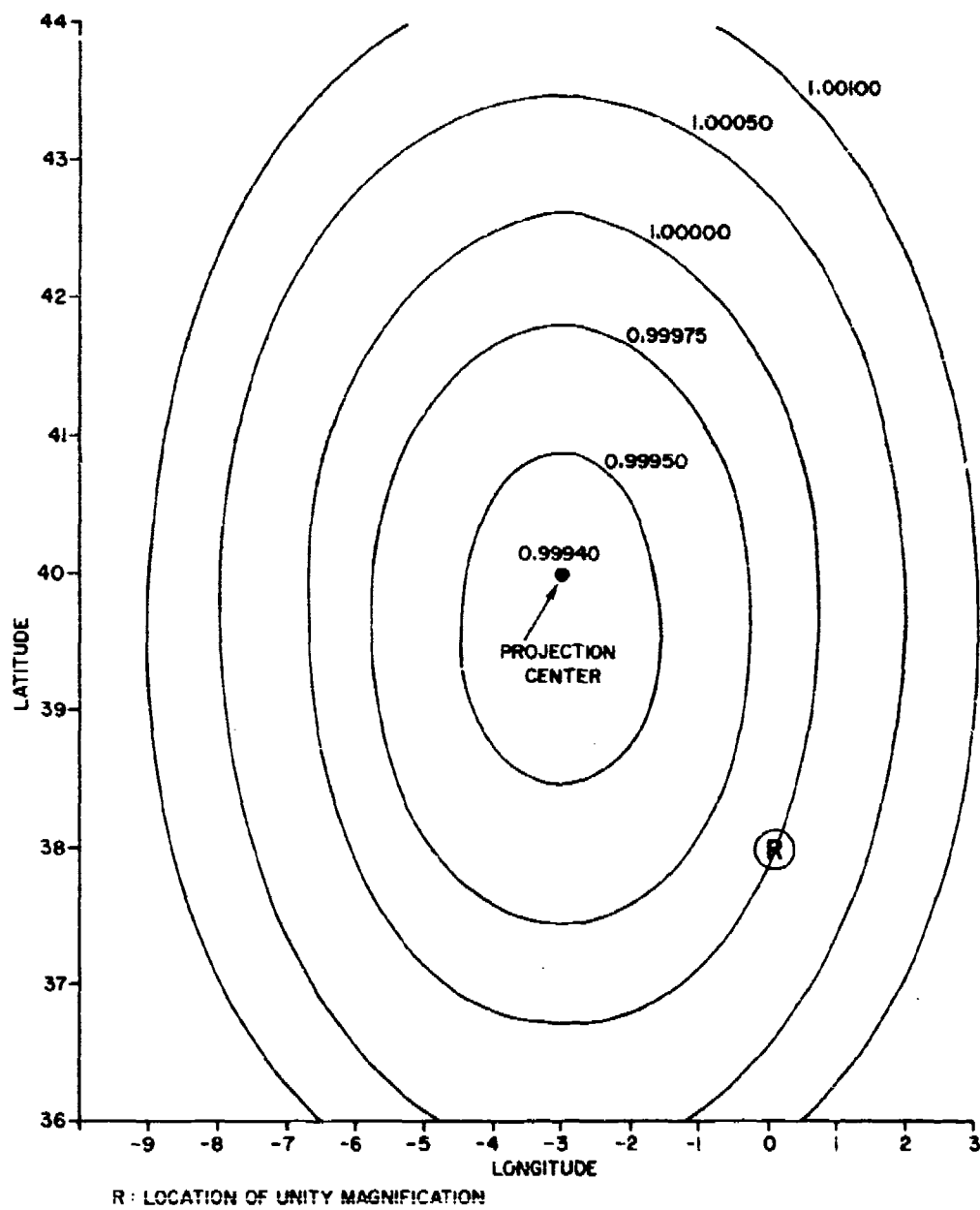
Figure II   MAGNIFICATION ; L,λ = 38°,0°

PC/R PROJECTION CENTER AND LOCATION OF UNITY MAGNIFICATION
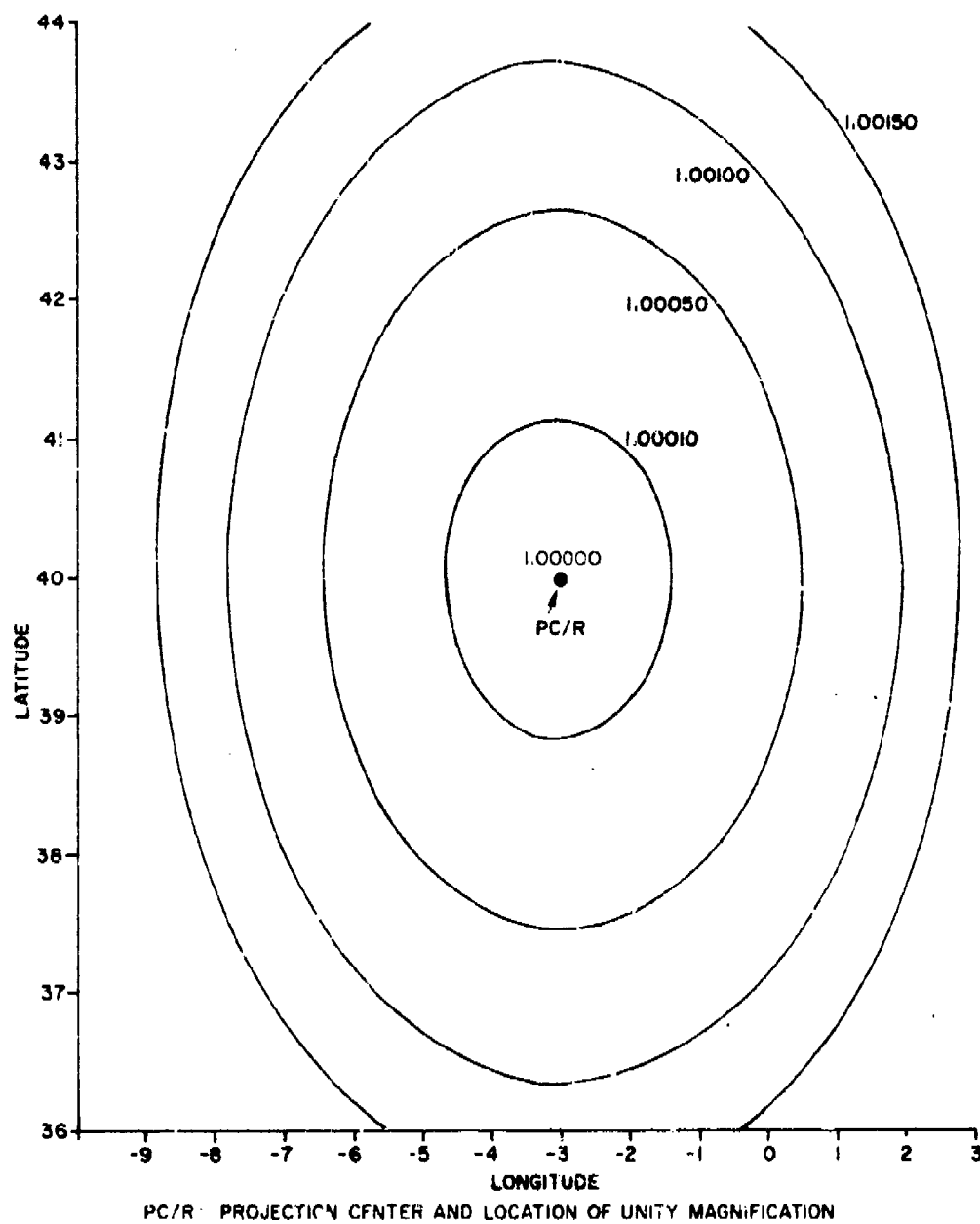
Figure 12 MAGNIFICATION ; L,λ= 40°,−3°

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values of the earth radii were approximately 3428 and 3440 miles respectively. To determine the impact of such a variation in the earth's radius, the stereographic coordinate conversion algorithm and the impact of the earth's radius therein will be considered.

Stereographic coordinate conversion consists of using slant range and azimuth data, the height of an aircraft above sea level and the elevation of the radar above sea level to obtain the stereographic projection of an aircraft's location onto the radar coordinate plane. From page 31 of Reference 3, a coordinate conversion equation which combines suitable accuracy with minimal processing requirements is:

$$R = \frac{[S^2 - (H-h)^2]^{\frac{1}{2}}}{1 + \frac{2}{E}} \tag{17}$$

where:

R is the stereographic ground range in the radar plane.

S is the measured slant range.

H is the measured height of the aircraft above sea level.

h is the elevation of the site above sea level.

E is the radius of the earth.

To determine the variation in the ground range R that results from using site dependent earth radii, equation (17) has been evaluated for several values of the numerator while E assumes the minimum and maximum values for our hypothetical area of coverage. Table IV shows the impact of using the minimum and maximum earth radii on the computation of ground range.

Table IV

Ground Range Versus Earth Radius

R based on:

Numerator (Mi)	$E_{min} = 3428$	$E_{max} = 3440$
225	224.8688	224.8693
100	99.9417	99.9419
30	29.9825	29.9826

Examination of Table IV shows that there is no essential difference in the calculated ground range due to the varying earth radii. Therefore, the earth radius as derived from equation (16) and which yields a unity magnification contour at a location approximately midway between the projection center and the maximum extent of radar coverage should also be used in the coordinate conversion algorithm.

SECTION IV

SUMMARY OF STEREOGRAPHIC PROJECTION ALGORITHM

INTRODUCTION

This section of the report reiterates the more important results that have been indicated herein. Additionally, a summary of the stereographic conversion and transformation equations is presented. The conversion algorithms have been developed in Reference 3 and the transformation equations have been derived herein.

ORIGIN OF COMMON COORDINATE PLANE

The latitude and longitude of the origin of the common coordinate plane are stored parameters that are required for stereographic projection of radar data. This origin should be chosen proximate to the center of the geographical area of interest to equalize the mapping distortion about this location and to minimize the overall distortion.

CHOICE OF EARTH'S RADIUS

Stereographic projection requires the mapping of points on or above a designated location on the earth ellipsoid onto a sphere which is called the conformal sphere. The radius of this sphere is a stored parameter that is required by both the stereographic transformation and conversion algorithms.

Having selected the origin of the common coordinate plane as described above and having selected a particular latitude and longitude through which a unity magnification contour is desired, the radius of the conformal sphere should be calculated from equation (18). It is recommended that the unity magnification contour should be located approximately midway between the origin of the common coordinate plane and the maximal extent of the radar coverage in order to decrease the registration error between observations of the same aircraft by different radars. The radius of the conformal sphere that is obtained from equation (18) should be used in both the stereographic conversion and transformation algorithms.

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$$E = \frac{N \cos L \,[1 + \sin \phi \, \sin \phi_o + \cos \phi \, \cos \phi_o \, \cos (\lambda - \lambda_o)]}{2 \cos \phi} \qquad (18)$$

where:

E   is the radius of the conformal sphere.

N   is $Eq(1-\varepsilon^2 \sin^2 L)^{-\frac{1}{2}}$ where Eq = 3444.054 miles is the earth's equatorial radius and where ε^2 = .00672267 is the earth's eccentricity squared.

L   is the geographic latitude of a designated location. This is the angle which a normal to the surface of an ellipsoid makes with the equatorial plane as shown in Figure 4.

ϕ, λ is the conformal latitude and longitude of a point through which the unity magnification contour is desired. ϕ is obtained via equation (19).

ϕ_o, λ_o is the conformal latitude and longitude of the origin of the common coordinate plane. ϕ_o is obtained via equation (19).

CONFORMAL AND GEOGRAPHIC LATITUDE

The conformal latitude may be obtained from the geographic latitude according to the relationship:

$$\tan \left[\left(\frac{\pi}{4} \right) + \left(\frac{\phi}{2} \right) \right] = \tan \left[\left(\frac{\pi}{4} \right) + \left(\frac{L}{2} \right) \right] \cdot \left[\frac{1 - \varepsilon \sin L}{1 + \varepsilon \sin L} \right]^{\varepsilon/2} \qquad (19)$$

where:

ϕ is the conformal latitude of a point, the geographic latitude of which is L.

ε = .08199189 is the earth's eccentricity.

For computational convenience, the conformal latitude ϕ may be obtained from the geographic latitude L by a series approximation to equation (19).

STEREOGRAPHIC COORDINATE CONVERSION

Slant range, azimuth and height data should be projected into the radar tangent plane as described in Reference 3. When height data is available the stereographic ground range R, the distance from the radar to the aircraft in the radar plane, should be calculated as follows:

$$R = \frac{[S^2 - (H-h)^2]^{\frac{1}{2}}}{1 + (2/E)} \tag{20}$$

When height data is unavailable, the stereographic ground range R should be calculated according to:

$$R = 1.0025 \ S - 0.65 \tag{21}$$

where:

   R   is the stereographic ground range.

   S   is the measured slant range.

   H   is the measured aircraft altitude above sea level.

   h   is the radar site elevation above sea level.

   E   is the radius of the conformal sphere from equation (18).

RADAR SITE COORDINATES

The rectangular coordinates of a radar site's location or any other point on the earth's surface relative to the origin of the common coordinate plane should be calculated from equations (22). Positive values of U_r and V_r correspond to locations that are east and north of the origin of the common coordinate plane.

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$$U_r = 2E \frac{\text{Sin}\ (\lambda - \lambda_o)\ \text{Cos}\ \phi}{1 + \text{Sin}\ \phi\ \text{Sin}\ \phi_o + \text{Cos}\ \phi\ \text{Cos}\ \phi_o\ \text{Cos}\ (\lambda - \lambda_o)}$$

$$V_r = 2E \frac{\text{Sin}\ \phi\ \text{Cos}\ \phi_o - \text{Cos}\ \phi\ \text{Sin}\ \phi_o\ \text{Cos}\ (\lambda - \lambda_o)}{1 + \text{Sin}\ \phi\ \text{Sin}\ \phi_o + \text{Cos}\ \phi\ \text{Cos}\ \phi_o\ \text{Cos}\ (\lambda - \lambda_o)}$$

(22)

where:

U_r, V_r are the rectangular coordinates of the point designated

by (L, λ) in the common coordinate plane.

ϕ, λ are the conformal latitude and longitude of the point

to be projected.

ϕ_o, λ_o are the conformal latitude and longitude of the origin

of the common coordinate plane.

E is the radius of the conformal sphere from equation (18).

Latitudes north of the equator are defined as positive whereas latitudes south of the equator are negative. Longitudes east of the prime meridian are defined as positive and those west of the prime meridian are negative.

ANGULAR ADJUSTMENT OF RADAR DATA

Radar data, which are measured relative to true north at the radar site location, should be adjusted so that the azimuth is relative to north at the origin of the common coordinate system. The angular adjustment β is defined as follows:

$$\beta = \text{Tan}^{-1} \left[\frac{(\text{Sin}\ \phi + \text{Sin}\ \phi_o)\ \text{Sin}\ (\lambda_o - \lambda)}{\text{Cos}\ \phi\ \text{Cos}\ \phi_o + (1 + \text{Sin}\ \phi\ \text{Sin}\ \phi_o)\ \text{Cos}\ (\lambda - \lambda_o)} \right]$$

(23)

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where:

ϕ, λ are the conformal latitude and longitude of the location of the radar site.

ϕ_o, λ_o are the conformal latitude and longitude of the origin of the common coordinate plane.

An azimuth datum θ, which is measured relative to true north at the radar site, should be adjusted to true north at the origin of the common coordinate system as follows:

$$\theta' = \theta + \beta$$

TRANSFORMATION FROM RADAR TO COMMON COORDINATES

The transformation from the radar coordinate plane to the common coordinate plane should be accomplished by an approximation to the stereographic transformation equation.

When aircraft height information is available, the rectangular coordinates of an aircraft with respect to the common coordinate origin should be calculated as follows:

$$U = U_r + K \left\{ R \sin (\theta+\beta) + \frac{R^2 |W_r| \sin [2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

$$V = V_r + K \left\{ R \cos (\theta+\beta) + \frac{R^2 |W_r| \cos [2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

(24)

where:

U_r, V_r are the rectangular coordinates of the radar site in the common coordinate plane.

R is the stereographic ground range in the radar plane from equation (20).

$$K = 1 + |W_r|^2/4E^2$$

$$|W_r| = [U_r^2 + V_r^2]^{\frac{1}{2}}$$

$$\gamma = \text{Tan}^{-1} (U_r/V_r)$$

Equations (24) may be calculated by the following expressions which preserve computational accuracy while minimizing the processing requirements.

$$U = U_r + T \left\{ A \sin (\theta+\beta) + \frac{A^2|W_r| \sin [2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

$$V = V_r + T \left\{ A \cos (\theta+\beta) + \frac{A^2|W_r| \cos [2(\theta+\beta) - \gamma]}{4E^2} \right\}$$

(25)

where:

$$T = \frac{K}{1 + \frac{2}{E}}$$

$$A = [S^2 - (H-h)^2]^{\frac{1}{2}}$$

When aircraft height information is unavailable, the rectangular coordinates of an aircraft with respect to the common coordinate origin should be calculated from:

$$U = U_r + K R \sin (\theta+\beta)$$

$$V = V_r + K R \cos (\theta+\beta)$$

(26)

where:

R   is the stereographic ground range in the radar plane from equation (21).

APPENDIX A

THE STEREOGRAPHIC TRANSFORMATION EQUATION

From pages 86 and 133 of Reference 5, a point on the surface of a spherical earth, the latitude of which is L and the longitude of which is λ, is stereographically projected onto a plane with an origin at a latitude of L_o and a longitude of λ_o by equation (A-1). The resultant point in the plane of projection is designated by Z and its rectangular components are X and Y.

$$Z = X + iY = 2Ei \frac{\exp[\frac{1}{2}(\tau-i\lambda-\tau_o+i\lambda_o)] - \exp[-\frac{1}{2}(\tau-i\lambda-\tau_o+i\lambda_o)]}{\exp[\frac{1}{2}(\tau-i\lambda+\tau_o+i\lambda_o)] + \exp[-\frac{1}{2}(\tau-i\lambda+\tau_o+i\lambda_o)]} \tag{A-1}$$

where:

$$e^\tau = \text{Tan}(\pi/4 + L/2)$$

$$e^{\tau_o} = \text{Tan}(\pi/4 + L_o/2)$$

L = latitude of the point to be projected

L_o = latitude of the coordinate origin

λ = longitude of the point to be projected

λ_o = longitude of the coordinate origin

E = radius of a spherical earth

Substituting $A = \frac{1}{2}(\tau-i\lambda)$, $B = \frac{1}{2}(-\tau_o+i\lambda_o)$ and $C = \frac{1}{2}(\tau_o+i\lambda_o)$, $D = \frac{1}{2}(-\tau_1+i\lambda_1)$ and $F = \frac{1}{2}(\tau_1+i\lambda_1)$, Z may be expressed as:

$$Z = 2Ei \frac{e^A e^B - e^{-A} e^{-B}}{e^A e^C + e^{-A} e^{-C}} = 2Ei \frac{e^B - e^{-2A} e^{-B}}{e^C + e^{-2A} e^{-C}} \tag{A-2}$$

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Solving equation (A-2) for exp $(\tau - i\lambda)$, we obtain

$$\exp(2A) = \exp(\tau - i\lambda) = \frac{Z + 2Ei \exp(\tau_0)}{2Ei \exp(i\lambda_0) - Z \exp(\tau_0 + i\lambda_0)} \qquad (A-3)$$

Let us now stereographically project the point (L,λ) onto a plane with an origin of (L_1, λ_1). The resultant point in the plane of projection is designated by W and its rectangular coordinates are U and V. The form of W will be identical to equation (A-1) except that τ_0 will be replaced by τ_1 and λ_0 will be replaced by λ_1. The point (L_1, λ_1) may be considered as the origin of the common coordinate plane and the point (L_0, λ_0) may be regarded as the origin of the radar plane. Rewriting equation (A-1) as suggested, W is:

$$W = 2Ei \frac{e^A e^D - e^{-A} e^{-D}}{e^A e^F + e^{-A} e^{-F}} = 2Ei \frac{e^{(D-F)} - e^{-2A} e^{-(D+F)}}{1 + e^{-2A} e^{-2F}} \qquad (A-4)$$

Substituting from (A-3) into (A-4) and performing the requisite manipulations, W may be expressed as:

$$2Ei \left\{ \frac{Z \exp(-\tau_1) + 2Ei \exp(\tau_0 - \tau_1) - 2Ei \exp[i(\lambda_0 - \lambda_1)] + Z \exp[\tau_0 + i(\lambda_0 - \lambda_1)]}{Z + 2Ei \exp(\tau_0) + 2Ei \exp[-\tau_1 + i(\lambda_0 - \lambda_1)] - Z \exp[(\tau_0 - \tau_1) + i(\lambda_0 - \lambda_1)]} \right\}$$

$$= \frac{Z\{\exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)]\} + 2Ei\{\exp(-\tau_1) - \exp[-\tau_0 + i(\lambda_0 - \lambda_1)]\}}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)] - \dfrac{Z}{4E^2} 2Ei\{\exp(-\tau_0) - \exp[-\tau_1 + i(\lambda_0 - \lambda_1)]\}}$$

Therefore, W may be written as:

$$W = \frac{Z \left\{ \dfrac{\exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)]}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)]} \right\} + 2Ei \left\{ \dfrac{\exp(-\tau_1) - \exp[-\tau_0 + i(\lambda_0 - \lambda_1)]}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)]} \right\}}{1 - \dfrac{Z}{4E^2} 2Ei \left\{ \dfrac{\exp(-\tau_0) - \exp[-\tau_1 + i(\lambda_0 - \lambda_1)]}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)]} \right\}} \qquad (A-5)$$

Equation (A-5) is the stereographic transformation equation. We will now examine this equation term by term in order to express it in a simpler form.

The fraction by which Z is multiplied in the numerator of equation (A-5) may be written as:

$$\frac{\exp\,[i(\lambda_0-\lambda_1)] + \exp\,[-(\tau_0+\tau_1)]}{1 + \exp\,[-(\tau_0+\tau_1) + i(\lambda_0-\lambda_1)]} = \frac{(e^\tau + \text{Cos }\Delta) + i\,\text{Sin }\Delta}{(1 + e^\tau\,\text{Cos }\Delta) + ie^\tau\,\text{Sin }\Delta} \tag{A-6}$$

where:

$$\tau = -(\tau_0 + \tau_1)$$

$$\Delta = (\lambda_0 - \lambda_1)$$

$$e^{\tau_j} = \text{Tan }[\tfrac{1}{2}(\pi/2 + L_j] = \left[\frac{1 + \text{Sin }L_j}{\text{Cos }L_j}\right] = \left[\frac{\text{Cos }L_j}{1 - \text{Sin }L_j}\right]$$

$$e^{-\tau_j} = \text{Tan }[\tfrac{1}{2}(\pi/2 - L_j] = \left[\frac{1 - \text{Sin }L_j}{\text{Cos }L_j}\right] = \left[\frac{\text{Cos }L_j}{1 + \text{Sin }L_j}\right]$$

$$j = 0 \text{ or } 1$$

Multiplying the numerator and denominator of (A-6) by its complex conjugate, and performing the necessary algebra, (A-6) may be expressed as:

$$\frac{2e^\tau + e^{2\tau}\,\text{Cos }\Delta + \text{Cos }\Delta + i[\text{Sin }\Delta\,(1-e^{2\tau})]}{1 + 2e^\tau\,\text{Cos }\Delta + e^{2\tau}} \tag{A-7}$$

Calculating the real (R) and imaginary (I) portions of expression (A-7) via the above definitions of $\exp\,(\tau_j)$ and $\exp\,(-\tau_j)$, R and I may be written as:

$$R = \frac{\text{Cos }L_0\,\text{Cos }L_1 + (1 + \text{Sin }L_0\,\text{Sin }L_1)\,\text{Cos }(\lambda_0-\lambda_1)}{1 + \text{Sin }L_0\,\text{Sin }L_1 + \text{Cos }L_0\,\text{Cos }L_1\,\text{Cos }(\lambda_0-\lambda_1)}$$

$$I = \frac{(\text{Sin }L_0 + \text{Sin }L_1)\,\text{Sin }(\lambda_0-\lambda_1)}{1 + \text{Sin }L_0\,\text{Sin }L_1 + \text{Cos }L_0\,\text{Cos }L_1\,\text{Cos }(\lambda_0-\lambda_1)}$$

$$\tag{A-8}$$

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Although tedious to verify, it may be shown that R^2+I^2 equals unity. Consequently, (A-6) may be expressed in terms of its polar coordinates as follows:

$$\frac{\exp\ [-(\tau_o + \tau_1)] + \exp\ [i(\lambda_o-\lambda_1)]}{1 + \exp\ [-(\tau_o + \tau_1) + i(\lambda_o-\lambda_1)]} = [R^2+I^2]^{\frac{1}{2}}\ e^{-i\beta} = e^{-i\beta}$$

where   (A-9)

$$\beta = \text{Tan}^{-1}\left[\frac{(\text{Sin } L_o + \text{Sin } L_1)\ \text{Sin }(\lambda_1-\lambda_o)}{\text{Cos } L_o\ \text{Cos } L_1 + (1 + \text{Sin } L_o\ \text{Sin } L_1)\ \text{Cos }(\lambda_o-\lambda_1)}\right]$$

and

L_o,λ_o are the latitude and longitude of the origin of the radar coordinate plane.

L_1,λ_1 are the latitude and longitude of the origin of the common coordinate plane.

From consideration of the definition of β, it will be noted that a positive β angle is measured in a counterclockwise direction from the R axis. Having reduced (A-6) to $e^{-i\beta}$, the leftmost term in the numerator of equation (A-5) may be expressed by $Ze^{-i\beta}$.

The right side of the numerator of equation (A-5) is:

$$2Ei\ \frac{\exp\ (-\tau_1) - \exp\ [-\tau_o + i(\lambda_o-\lambda_1)]}{1 + \exp\ [-(\tau_o + \tau_1) + i(\lambda_o-\lambda_1)]}$$

$$= 2Ei\ \frac{\exp(\tau_o-i\lambda_o) - \exp(\tau_1-i\lambda_1) + \exp(-\tau_1-i\lambda_1) - \exp(-2i\lambda_1-\tau_o+i\lambda_o)}{[\exp(-i\lambda_1) + \exp(\tau_o+\tau_1-i\lambda_o)]\ \{1 + \exp\ [-(\tau_o+\tau_1) + i(\lambda_o-\lambda_1)]\}}$$

$$= 2Ei\ \frac{\exp\ (\tau_o-i\lambda_o) - \exp\ (\tau_1-i\lambda_1)}{\exp\ (-i\lambda_1) + \exp\ (\tau_o+\tau_1-i\lambda_o)}$$

$$= 2Ei\ \frac{\exp\ [\frac{1}{2}(\tau_o-i\lambda_o-\tau_1+i\lambda_1)] - \exp\ [-\frac{1}{2}(\tau_o-i\lambda_o-\tau_1+i\lambda_1)]}{\exp\ [\frac{1}{2}(\tau_o-i\lambda_o+\tau_1+i\lambda_1)] + \exp\ [-\frac{1}{2}(\tau_o-i\lambda_o+\tau_1+i\lambda_1)]} = W_r \qquad (A-10)$$

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Therefore, the right side of the numerator of equation (A-5) is W_r, which represents the origin of the radar plane projected onto the common coordinate plane. The entire numerator of equation (A-5) is $(Ze^{-i\beta} + W_r)$.

The fraction by which $(-Z/4E^2)$ is multiplied in the denominator of equation (A-5) is:

$$2Ei \; \frac{\exp(-\tau_0) - \exp[-\tau_1 + i(\lambda_0 - \lambda_1)]}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)]} = 2Ei \; \frac{\exp(-\tau_0) - \exp[-\tau_1 + i(\lambda_0 - \lambda_1)]}{\exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)]}$$

$$X \; \frac{\exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)]}{1 + \exp[-(\tau_0 + \tau_1) + i(\lambda_0 - \lambda_1)]}$$

which from expression (A-9) is:

$$\left\{ 2Ei \; \frac{\exp(-\tau_0) - \exp[-\tau_1 + i(\lambda_0 - \lambda_1)]}{\exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)]} \right\} e^{-i\beta} \tag{A-11}$$

The term within the parenthesis may be expressed as:

$$-2Ei \; \frac{\exp[\tau_0 + i(2\lambda_0 - \lambda_1)] + \exp(-\tau_1 + i\lambda_0) - \exp(\tau_1 + i\lambda_0) - \exp(-\tau_0 + i\lambda_1)}{[\exp(\tau_0 + \tau_1 + i\lambda_0) + \exp(i\lambda_1)] \{ \exp[i(\lambda_0 - \lambda_1)] + \exp[-(\tau_0 + \tau_1)] \}}$$

$$= -2Ei \; \frac{\exp(\tau_0 + i\lambda_0) - \exp(\tau_1 + i\lambda_1)}{\exp(\tau_0 + \tau_1 + i\lambda_0) + \exp(i\lambda_1)}$$

$$= -2Ei \; \frac{\exp[\frac{1}{2}(\tau_0 + i\lambda_0 - \tau_1 - i\lambda_1)] - \exp[-\frac{1}{2}(\tau_0 + i\lambda_0 - \tau_1 - i\lambda_1)]}{\exp[\frac{1}{2}(\tau_0 + i\lambda_0 + \tau_1 - i\lambda_1)] + \exp[-\frac{1}{2}(\tau_0 + i\lambda_0 + \tau_1 - i\lambda_1)]} = \overline{W}_r \tag{A-12}$$

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where $\overline{W}_r = U_r - iV_r$ is the complex conjugate of W_r. Therefore, the fraction by which $(-Z/4E^2)$ is multiplied in the denominator of equation (A-5) is $\overline{W}_r\, e^{-i\beta}$.

Substituting the results of expressions (A-9) through (A-12) into equation (A-5) yields:

$$W = \frac{Ze^{-i\beta} + W_r}{1 - \dfrac{Z\,\overline{W}_r\, e^{-i\beta}}{4E^2}} \tag{A-13}$$

APPENDIX B

PROJECTING THE EARTH'S SURFACE ONTO A PLANE

This Appendix derives for a spherical earth the equations which stereographically project points on the surface of the earth onto a plane of rectangular coordinates. Figure B-1 depicts the geometry for a spherical triangle with the origin of the coordinates at latitude L_o and longitude λ_o. The point to be projected has coordinates L,λ.

In Figure B-1, N and E are the directions of north and east respectively in the plane of projection. The line OP' is the arc OP projected into the plane. The angle γ in the plane corresponds to the spherical angle ZOP. $\emptyset$ is the angle between the radii drawn from the earth's center to the origin of the coordinate system and to the point to be projected.

From the law of cosines for the spherical triangle OPZ, we obtain:

$$\text{Cos PO} = \text{Cos PZ Cos OZ} + \text{Sin PZ Sin OZ Cos PZO}$$

$$\text{Cos } \emptyset = \text{Sin L Sin } L_o + \text{Cos L Cos } L_o \text{ Cos } (\lambda-\lambda_o) \tag{B-1}$$

From spherical triginometry, the relationship between two angles and three sides is:

$$\text{Sin PO Cos ZOP} = \text{Cos PZ Sin ZO} - \text{Cos ZO Sin PZ Cos PZO}$$

$$\text{Sin } \emptyset \text{ Cos } \gamma = \text{Sin L Cos } L_o - \text{Sin } L_o \text{ Cos L Cos } (\lambda-\lambda_o) \tag{B-2}$$

From the law of sines:

$$\frac{\text{Sin PO}}{\text{Sin PZO}} = \frac{\text{Sin PZ}}{\text{Sin ZOP}} = \frac{\text{Sin } \emptyset}{\text{Sin } (\lambda-\lambda_o)} = \frac{\text{Cos L}}{\text{Sin } \gamma}$$

Therefore, $\text{Sin } \emptyset \text{ Sin } \gamma = \text{Sin } (\lambda-\lambda_o) \text{ Cos L}$ (B-3)

From equation (2) of Reference 3, the stereographic ground range designated $|W_r|$ herein, may be obtained from the relationship:

$$|W_r| = 2E \text{ Tan } (\emptyset/2) = 2E \frac{\text{Sin } \emptyset}{1 + \text{Cos } \emptyset}$$
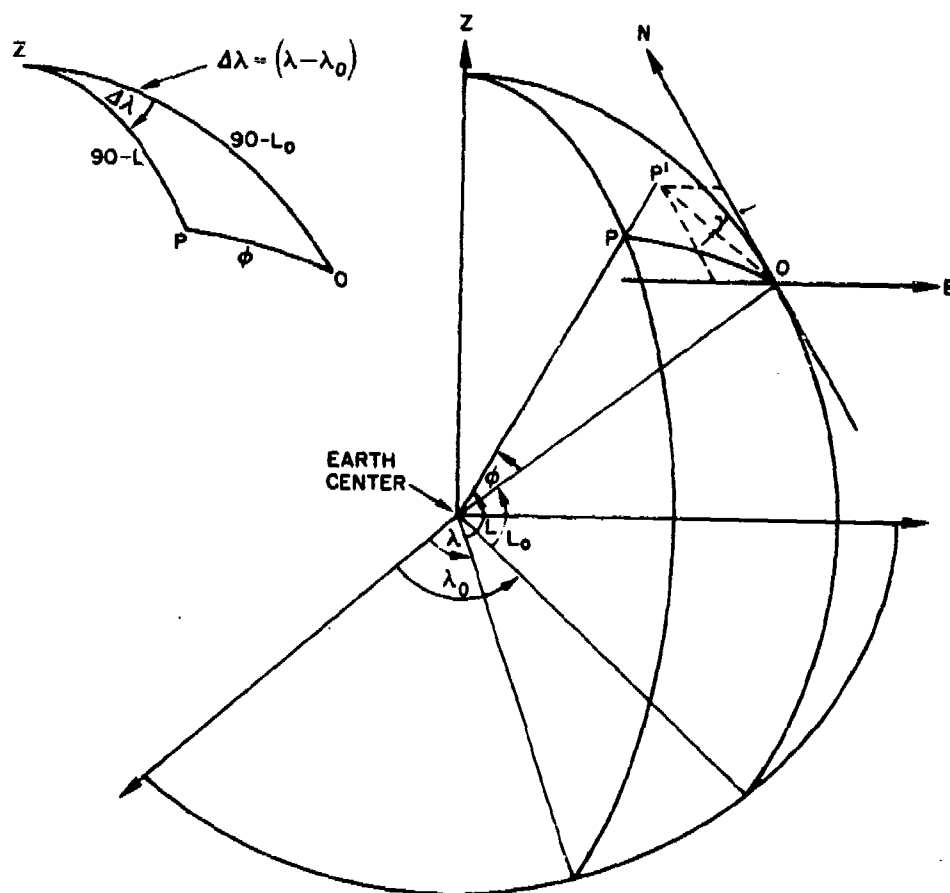
Figure B-1 PROJECTING EARTH ONTO PLANE

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The U_r, V_r components of the projected point W_r are obtained from:

$$U_r = |W_r| \sin \gamma = \frac{2E \sin \emptyset \sin \gamma}{1 + \cos \emptyset}$$

(B-4)

$$V_r = |W_r| \cos \gamma = \frac{2E \sin \emptyset \cos \gamma}{1 + \cos \emptyset}$$

Substituting from equations (B-1), (B-2), and (B-3) into (B-4), we obtain:

$$U_r = \frac{2E \sin (\lambda - \lambda_o) \cos L}{1 + \sin L \sin L_o + \cos L \cos L_o \cos (\lambda - \lambda_o)}$$

(B-5)

$$V_r = \frac{2E[\sin L \cos L_o - \sin L_o \cos L \cos (\lambda - \lambda_o)]}{1 + \sin L \sin L_o + \cos L \cos L_o \cos (\lambda - \lambda_o)}$$

where:

L, λ are the latitude and longitude of the point to be projected onto the coordinate plane.

L_o, λ_o are the latitude and longitude of the origin of the plane of rectangular coordinates.

U_r, V_r are the rectangular coordinates of the point L, λ in the plane of projection.

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